Answers

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Non-exact numerical answers should be given correct to three significant figures (or one decimal place for angles in degrees) unless a different level of accuracy is specified in the question. You should avoid rounding figures until reaching your final answer.

Chapter 1

? (Page 1)

4, 1, 0

? (Page 4)

When subtracting numbers, the order in which the numbers appear is important – changing the order changes the answer, for example $3 - 6 \neq 6 - 3$. So subtraction of numbers is not commutative.

The grouping of the numbers is also important, for example $(13 - 5) - 2 \neq 13 - (5 - 2)$. Therefore subtraction of numbers is not associative.

Matrices follow the same rules for commutativity and associativity as numbers. Matrix addition is both commutative and associative, but matrix subtraction is not commutative or associative. This is true because addition and subtraction of each of the individual elements will determine whether the matrices are commutative or associative overall.

You can use more formal methods to prove these properties. For example, to show that matrix addition is commutative:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Addition of numbers is commutative.

Exercise 1A (Page 4)

1 (i) 3×2 (ii) 3×3 (iii) 1×2
(iv) 5×1 (v) 2×4 (vi) 3×2

2 (i) $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 2 & 12 \end{pmatrix}$

(iii) $\begin{pmatrix} -8 & 5 \\ -3 & 7 \end{pmatrix}$

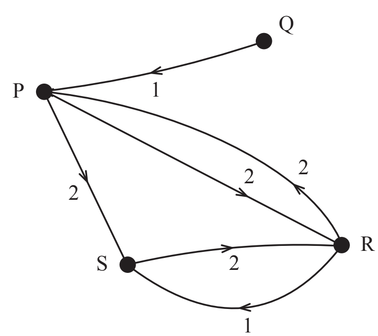
(iv) Non-conformable (v) $\begin{pmatrix} -3 & -9 & 14 \\ 0 & 0 & 4 \end{pmatrix}$

(vi) $\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$ (vii) $\begin{pmatrix} 9 & 7 & -17 \\ 10 & 5 & 28 \end{pmatrix}$

(viii) Non-conformable (ix) $\begin{pmatrix} -15 & 8 \\ -4 & 3 \end{pmatrix}$

3 (i) $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$

(iii)



4 $w = 2, x = -6, y = -2, z = 2$

5 $p = -1$ or 6, $q = \pm\sqrt{5}$

6 (i) $\begin{pmatrix} 1 & 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 7 & 5 \\ 0 & 1 & 0 & 3 & 3 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 & 1 & 10 & 7 \\ 0 & 0 & 4 & 2 & 10 \\ 3 & 1 & 1 & 11 & 8 \\ 1 & 2 & 1 & 8 & 6 \end{pmatrix}$

Answers

(ii) $\begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$

Stars 2 vs United 1
Cougars 2 vs Town 1
Cougars 1 vs United 1

7 (i) $\begin{pmatrix} 15 & 3 & 7 & 15 \\ 5 & 9 & 15 & -3 \\ 19 & 10 & 9 & 3 \end{pmatrix}$

The matrix represents the number of jackets left in stock after all the orders have been dispatched. The negative element indicates there was not enough of that type of jacket in stock to fulfil the order.

(ii) $\begin{pmatrix} 20 & 13 & 17 & 20 \\ 15 & 19 & 20 & 12 \\ 19 & 10 & 14 & 8 \end{pmatrix}$

(iii) $\begin{pmatrix} 12 & 30 & 18 & 0 \\ 6 & 18 & 24 & 36 \\ 30 & 0 & 12 & 18 \end{pmatrix}$

The assumption is probably not very realistic, as a week is quite a short time.

? (Page 9)

The dimensions of the matrices are **A** (3 × 3), **B** (3 × 2) and **C** (2 × 2). The conformable products are **AB** and **BC**. Both of these products would have dimension (3 × 2), even though the original matrices are not the same sizes.

Activity 1.1 (Page 10)

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ -20 & 4 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -8 & 4 \\ -1 & 6 \end{pmatrix}$$

These two matrices are not equal and so matrix multiplication is not usually commutative. There are some exceptions, for example if

$\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ then

$$\mathbf{CD} = \mathbf{DC} = \begin{pmatrix} 6 & 6 \\ -2 & -2 \end{pmatrix}.$$

Activity 1.2 (Page 10)

(i) $\mathbf{AB} = \begin{pmatrix} -6 & -1 \\ -20 & 4 \end{pmatrix}$

(ii) $\mathbf{BC} = \begin{pmatrix} -4 & -8 \\ 0 & -1 \end{pmatrix}$

(iii) $(\mathbf{AB})\mathbf{C} = \begin{pmatrix} -8 & -15 \\ -12 & -28 \end{pmatrix}$

(iv) $\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} -8 & -15 \\ -12 & -28 \end{pmatrix}$

(**AB**)**C** = **A**(**BC**) so matrix multiplication is associative in this case

To produce a general proof, use general matrices such as

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} i & j \\ k & l \end{pmatrix}.$$

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

$$\mathbf{BC} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} ei + fk & ej + fl \\ gi + hk & gj + hl \end{pmatrix}$$

and so

$$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

$$= \begin{pmatrix} aei + bgi + afk + bhk & aej + bgj + afl + bhl \\ cei + dgi + cfk + dhk & cej + dgj + cfl + dhl \end{pmatrix}$$

and

$$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} ei + fk & ej + fl \\ gi + hk & gj + hl \end{pmatrix}$$

$$= \begin{pmatrix} aei + afk + bgi + bhk & aej + afl + bgj + bhl \\ cei + dgi + cfk + dhk & cej + dgj + cfl + dhl \end{pmatrix}$$

Since (**AB**)**C** = **A**(**BC**) matrix multiplication is associative and the product can be written without brackets as **ABC**.

Exercise 1B (Page 11)

1 (i) (a) 3×3 (b) 1×3 (c) 2×3
(d) 2×4 (e) 2×1 (f) 3×5

(ii) (a) non-conformable
(b) 3×5 (c) non-conformable
(d) 2×3 (e) non-conformable

2 (i) $\begin{pmatrix} 21 & 6 \\ 31 & 13 \end{pmatrix}$ (ii) $(-30 \quad -15)$

(iii) $\begin{pmatrix} -54 \\ -1 \end{pmatrix}$

3 $\mathbf{AB} = \begin{pmatrix} 3 & -56 \\ 20 & -73 \end{pmatrix}$, $\mathbf{BA} = \begin{pmatrix} -25 & 8 \\ 28 & -45 \end{pmatrix}$

$\mathbf{AB} \neq \mathbf{BA}$ so matrix multiplication is non-commutative.

4 (i) $\begin{pmatrix} -7 & 26 \\ 2 & 34 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 25 \\ 16 & 22 \end{pmatrix}$

(iii) $\begin{pmatrix} 31 & 0 \\ 65 & 18 \end{pmatrix}$ (iv) $\begin{pmatrix} 26 & 37 & 16 \\ 14 & 21 & 28 \\ -8 & -11 & 2 \end{pmatrix}$

(v) non-conformable (vi) $\begin{pmatrix} 28 & -18 \\ 26 & 2 \\ 16 & 25 \end{pmatrix}$

5 $\begin{pmatrix} -38 & -136 & -135 \\ 133 & 133 & 100 \\ 273 & 404 & 369 \end{pmatrix}$

6 (i) $\begin{pmatrix} 2x^2 + 12 & -9 \\ -4 & 3 \end{pmatrix}$ (ii) $x = 2$ or 3

(iii) $\mathbf{BA} = \begin{pmatrix} 8 & 12 \\ 8 & 15 \end{pmatrix}$ or $\begin{pmatrix} 18 & 18 \\ 12 & 15 \end{pmatrix}$

7 (i) (a) $\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 7 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 16 & 15 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1024 & 1023 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

8 (i) $\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 4 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 2 & 1 & 5 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$

$\mathbf{M}^2$ represents the number of two-stage routes between each pair of resorts.

(iii) $\mathbf{M}^3$ would represent the number of three-stage routes between each pair of resorts.

9 (i) $\begin{pmatrix} 8 + 4x & -20 + x^2 \\ -8 + x & -3 - 3x \end{pmatrix}$

(ii) $x = -3$ or 4

(iii) $\begin{pmatrix} -4 & -11 \\ -11 & 6 \end{pmatrix}$ or $\begin{pmatrix} 24 & -4 \\ -4 & -15 \end{pmatrix}$

10 (i) $\begin{pmatrix} b \\ a \\ c \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} b \\ c \\ a \end{pmatrix}$

(iv) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} c \\ a \\ b \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The strands are back in the original order at the end of Stage 6.

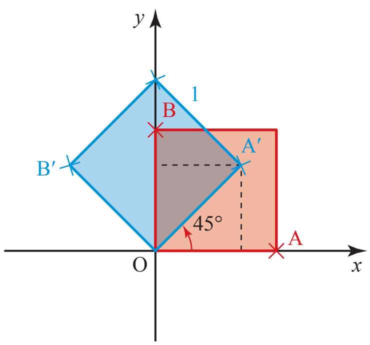
(Page 18)

The image of the unit vector $\begin{pmatrix}1\\0\end{pmatrix}$ is $\begin{pmatrix}a\\c\end{pmatrix}$ and the image of the unit vector $\begin{pmatrix}0\\1\end{pmatrix}$ is $\begin{pmatrix}b\\d\end{pmatrix}$.

The origin maps to itself.

Activity 1.3 (Page 18)

The diagram below shows the unit square with two of its sides along the unit vectors $\mathbf{i}$ and $\mathbf{j}$. It is rotated by 45° about the origin.



You can use trigonometry to find the images of the unit vectors **i** and **j**.

For A′, the x-coordinate satisfies $\cos 45 = \frac{x}{1}$ so $x = \cos 45 = \frac{1}{\sqrt{2}}$.

In a similar way, the y-coordinate of A′ is $\frac{1}{\sqrt{2}}$.

For B′, the symmetry of the diagram shows that the x-coordinate is $-\frac{1}{\sqrt{2}}$ and the y-coordinate is $\frac{1}{\sqrt{2}}$.

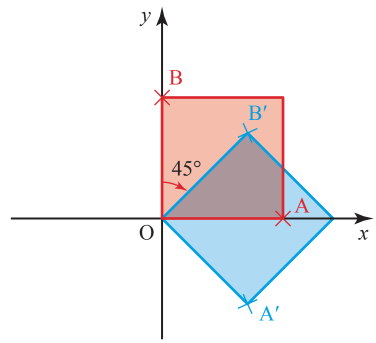
Hence, the image of $\begin{pmatrix}1\\0\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ and the image of $\begin{pmatrix}0\\1\end{pmatrix}$ is $\begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ and so the matrix representing an anticlockwise rotation of 45° about the origin is

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}.$$

Rotations of 45° clockwise about the origin and 135° anticlockwise about the origin are also represented by matrices involving $\pm\frac{1}{\sqrt{2}}$.

This is due to the symmetry about the origin.

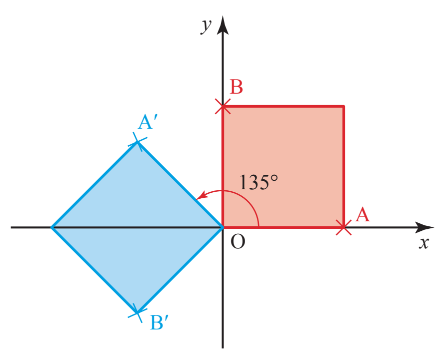
(i) The diagram for a 45° clockwise rotation about the origin is shown below.



The image of $\begin{pmatrix}1\\0\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$ and the image of $\begin{pmatrix}0\\1\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ and so the matrix representing an anticlockwise rotation of 45° about the origin is

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}.$$

(ii) The diagram for a 135° anticlockwise rotation about the origin is shown below.



The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ and so the matrix representing an anticlockwise rotation of 45° about the origin is

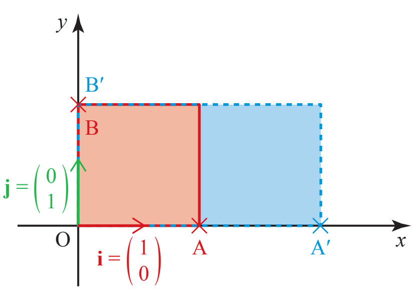
$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

? (Page 20)

The matrix for a rotation of $\theta°$ clockwise about the origin is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

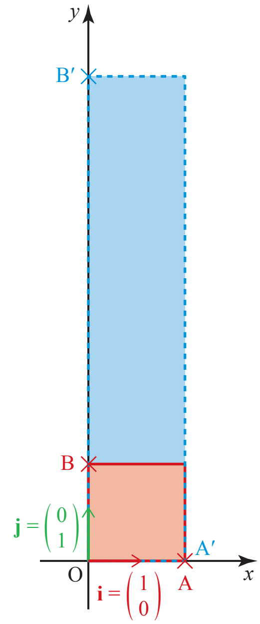
Activity 1.4 (Page 20)

(i) The diagram below shows the effect of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ on the unit vectors **i** and **j**.



You can see that the vector **i** has image $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and the vector **j** is unchanged. Therefore this matrix represents a stretch of scale factor 2 parallel to the x-axis.

(ii) The diagram below shows the effect of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ on the unit vectors **i** and **j**.



You can see that the vector **i** is unchanged and the vector **j** has image $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$. Therefore this matrix represents a stretch of scale factor 5 parallel to the y-axis.

The matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ represents a stretch of scale factor m parallel to the x-axis.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ represents a stretch of scale factor n parallel to the y-axis.

Activity 1.5 (Page 22)

Point A: $6 \div 2 = 3$

Point B: $6 \div 2 = 3$

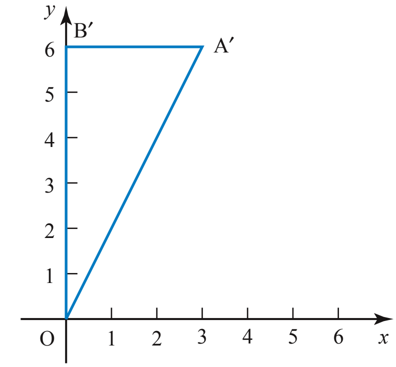
Point C: $3 \div 1 = 3$

Point D: $3 \div 1 = 3$

The ratio is equal to 3 for each point.

Exercise 1C (Page 24)

1 (i) (a)

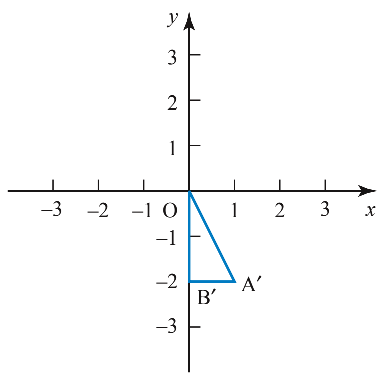


(b) $A' = (3, 6), B' = (0, 6)$

(c) $x' = 3x, y' = 3y$

(d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

(ii) (a)

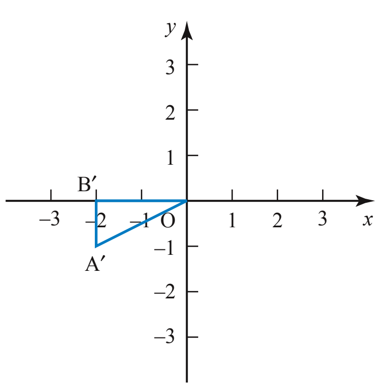


(b) $A' = (1, -2), B' = (0, -2)$

(c) $x' = x, y' = -y$

(d) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) (a)

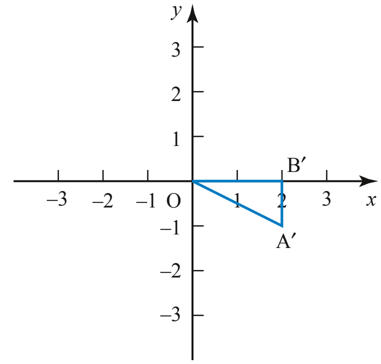


(b) $A' = (-2, -1), B' = (-2, 0)$

(c) $x' = -y, y' = -x$

(d) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(iv) (a)

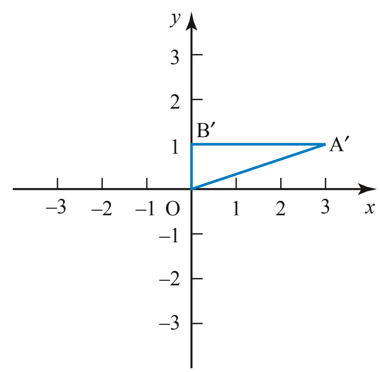


(b) $A' = (2, -1), B' = (2, 0)$

(c) $x' = y, y' = -x$

(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(v) (a)



(b) $A' = (3, 1), B' = (0, 1)$

(c) $x' = 3x, y' = \frac{1}{2}y$

(d) $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

2 (i) Reflection in the x-axis

(ii) Reflection in the line $y = -x$

(iii) Stretch of factor 2 parallel to the x-axis and stretch factor 3 parallel to the y-axis

(iv) Enlargement, scale factor 4, centre the origin

(v) Rotation of 90° clockwise (or 270° anticlockwise) about the origin

3 (i) Rotation of 60° anticlockwise about the origin

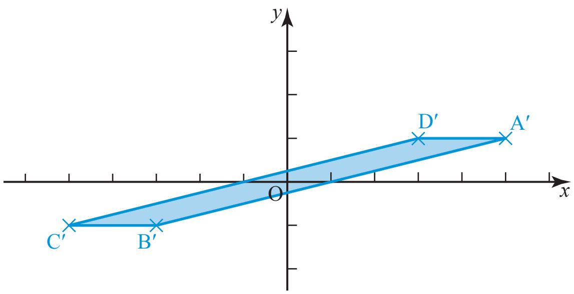
(ii) Rotation of 55° anticlockwise about the origin

(iii) Rotation of 135° clockwise about the origin

(iv) Rotation of 150° anticlockwise about the origin

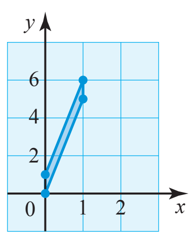
4 (i) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -3 & -5 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

so the transformed square would look like this:

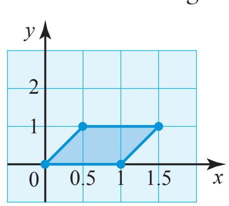


(ii) The transformation is a shear with the x-axis fixed and the point A(1, 1) has image A′(5, 1).

5 (i) (a) The image of the unit square has vertices (0, 0), (1, 5), (0, 1), (1, 6) as shown in the diagram below.



(b) The image of the unit square has vertices (0, 0), (1, 0), (0.5, 1), (1.5, 1) as shown in the diagram below.



(ii) Matrix **A** represents a shear with the y-axis fixed; the point (1, 1) has image (1, 6). **A** has shear factor 5.

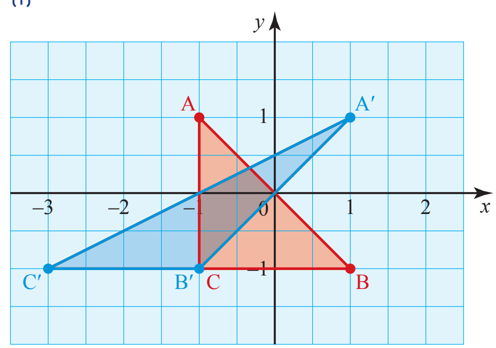
Matrix **B** represents a shear with the x-axis fixed; the point (1, 1) has image (1.5, 1).

B has shear factor 0.5.

6 (i) $A'(2\sqrt{3} - 1,\ 2)$ (ii) $\begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}$

7 A′(4, 5), B′(7, 9), C′(3, 4). The original square and the image both have an area of one square unit.

8 (i)



(ii) The gradient of A′C′ is $\frac{1}{2}$, which is the reciprocal of the top right-hand entry of the matrix **M**.

9 $(x, y) \rightarrow (x, x)$

The matrix for the transformation is $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

10 (i) Any matrix of the form $\begin{pmatrix} 5 & 0 \\ 0 & k \end{pmatrix}$ or $\begin{pmatrix} k & 0 \\ 0 & 5 \end{pmatrix}$.

If $k = 5$ the rectangle would be a square.

(ii) $\begin{pmatrix} \sqrt{2} & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{pmatrix}$,

$\begin{pmatrix} 1 & \sqrt{2} \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ \sqrt{2} & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 7 & \frac{3\sqrt{3}}{2} \\ 0 & \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{2} \\ 7 & \frac{3\sqrt{3}}{2} \end{pmatrix},$

$\begin{pmatrix} \frac{3\sqrt{3}}{2} & 7 \\ \frac{3}{2} & 0 \end{pmatrix}$ or $\begin{pmatrix} \frac{3}{2} & 0 \\ \frac{3\sqrt{3}}{2} & 7 \end{pmatrix}$

? (Page 27)

BA represents a reflection in the line $y = x$. The transformation **A** is represented by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the transformation B is represented by the matrix

$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The matrix product

$\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

This is the matrix which represents a reflection in the line $y = x$.

Activity 1.6 (Page 28)

(i) $\mathbf{P}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

(ii) $\mathbf{P}'' = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

$= \begin{pmatrix} pax + pby + qcx + qdy \\ rax + rby + scx + sdy \end{pmatrix}$

(iii) $\mathbf{U} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}$

and so

$\mathbf{UP} = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} pax + qcx + pby + qdy \\ rax + scx + rby + sdy \end{pmatrix}$.

Therefore **UP=P″**

? (Page 28)

AB represents 'carry out transformation **B** followed by transformation **A**.

(**AB**)**C** represents 'carry out transformation **C** followed by transformation **AB**, i.e. 'carry out **C** followed by **B** followed by **A**'.

BC represents 'carry out transformation **C** followed by transformation **B**'.

A(**BC**) represents 'carry out transformation **BC** followed by transformation **A**, i.e. carry out **C** followed by **B** followed by **A**'.

Activity 1.7 (Page 29)

(i) $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$

$\mathbf{B} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

(ii)
$\mathbf{BA} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\sin\theta\cos\phi - \cos\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$

(iii) $\mathbf{C} = \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}$

(iv) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

(v) A rotation through angle θ followed by rotation through angle ϕ has the same effect as a rotation through angle ϕ followed by angle θ.

Exercise 1D (Page 30)

1 (i) **A**: enlargement centre (0,0), scale factor 3
B: rotation 90° anticlockwise about (0,0)
C: reflection in the x-axis
D: reflection in the line $y = x$

(ii) (a) $\mathbf{BC} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, reflection in the line $y = x$

(b) $\mathbf{CB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, reflection in the line $y = -x$

(c) $\mathbf{DC} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, rotation 90° anticlockwise about (0, 0)

(d) $\mathbf{A}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$, enlargement centre (0, 0), scale factor 9

(e) $\mathbf{BCB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, reflection in the x-axis

(f) $\mathbf{DC}^2\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ returns the object to its original position

(iii) For example, $\mathbf{B}^4$, $\mathbf{C}^2$ or $\mathbf{D}^2$

2 (i) $\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \mathbf{Y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $\mathbf{XY} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, rotation of 180° about the origin

(iii) $\mathbf{YX} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(iv) When considering the effect on the unit vectors **i** and **j**, as each transformation only affects one of the unit vectors the order of the transformations is not important in this case.

3 (i) $\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) $\mathbf{PQ} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, reflection in the line $y = -x$

(iii) $\mathbf{QP} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(iv) The matrix **P** has the effect of making the coordinates of any point the negative of their original values,

i.e. $(x, y) \rightarrow (-x, -y)$

The matrix **Q** interchanges the coordinates,

i.e. $(x, y) \rightarrow (y, x)$

It does not matter what order these two transformations occur as the result will be the same

4 (i) $\begin{pmatrix} 8 & -4 \\ -3 & 12 \end{pmatrix}$ $\quad$ (ii) $(32, -33)$

5 Possible transformations are $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is a rotation of 90° clockwise about the origin, followed by

$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, which is a stretch of scale factor 3 parallel to the x-axis. The order of these is important as performing **A** followed by **B** leads to the matrix $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$. Could also have

$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. which represents a stretch of factor 3 parallel to the y-axis, followed by

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which represents a rotation of 90° clockwise about the origin; again the order is important.

6 (i) $\mathbf{PQ} = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$

(ii) $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection in the x-axis.

$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ represents a shear with the y-axis fixed; point B(1,1) has image (1,–4).

7 $\mathbf{X} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

A matrix representing a rotation about the origin has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and so the entries on the leading diagonal would be equal. That is not true for matrix **X** and so this cannot represent a rotation.

8 (i) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(ii) A reflection in the x-axis and a stretch of scale factor 5 parallel to the x-axis

(iii) $\begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$

Reflection in the x-axis; stretch of scale factor 5 parallel to the x-axis; stretch of scale factor 2 parallel to the y-axis. The outcome of these three transformations would be the same regardless of the order in which they are applied. There are six different possible orders.

(iv) $\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

9 (i) $\begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1 \end{pmatrix}$

(iv) $\begin{pmatrix} 1+\frac{R_1}{R_2} & -R_1 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$

The effect of Type B followed by Type A is different to that of Type A followed by Type B.

10 $a = \sqrt{\frac{\sqrt{2}+2}{4}}$ and $b = \sqrt{\frac{1}{2\left(\sqrt{2}+2\right)}}$

D represents an anticlockwise rotation of 22.5° about the origin.

By comparison to the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for an anticlockwise rotation of θ about the origin, a and b are the exact values of $\cos 22.5°$ and $\sin 22.5°$ respectively.

11 (i) $\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

(ii) $\mathbf{QP} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, which represents a rotation of 60° anticlockwise about the origin.

(iii) $\mathbf{PQ} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, which represents a rotation of 60° clockwise about the origin.

12 A reflection in a line followed by a second reflection in the same line returns a point to its original position.

? (Page 33)

In a reflection, all points on the mirror line map to themselves.

In a rotation, only the centre of rotation maps to itself.

Exercise 1E (Page 35)

1 (i) Points of the form $(\lambda, -2\lambda)$

(ii) (0, 0)

(iii) Points of the form $(\lambda, -3\lambda)$

(iv) Points of the form $(2\lambda,\ 3\lambda)$

2 (i) x-axis, y-axis, lines of the form $y = mx$

(ii) x-axis, y-axis, lines of the form $y = mx$

(iii) no invariant lines

(iv) $y = x$, lines of the form $y = -x + c$

(v) $y = -x$, lines of the form $y = x + c$

(vi) x-axis, lines of the form $y = c$

3 (i) Any points on the line $y = \frac{1}{2}x$, for example (0, 0), (2, 1) and (3, 1.5)

(ii) $y = \frac{1}{2}x$

(iii) Any line of the form $y = -2x + c$

(iv) Using the method of Example 1.12 leads to the equations

$2m^2 + 3m - 2 = 0 \Rightarrow m = 0.5$ or -2

$(4 + 2m)c = 0 \Rightarrow m = -2$ or $c = 0$

If $m = 0.5$ then $c = 0$ so $y = \frac{1}{2}x$ is invariant.

If $m = -2$ then c can take any value and so $y = -2x + c$ is an invariant line.

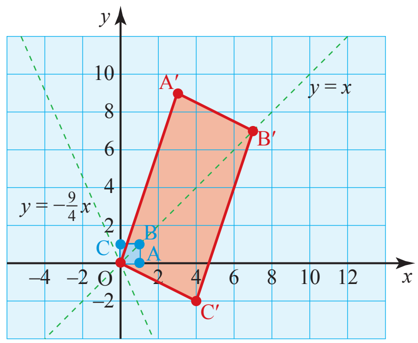
4 (i) Solving $\begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ leads to the equations $y = -\frac{3x}{11}$ and $y = -\frac{11x}{3}$.

The only point that satisfies both of these is (0, 0).

(ii) $y = x$ and $y = -x$

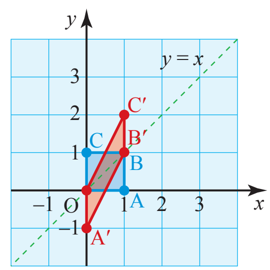
5 (i) $y = x$, $y = -\frac{9}{4}x$

(ii)



6 (i) $y = x$ (ii) $y = x$

(iii)



9 (i) $x' = x + a,\ y' = y + b$

(iii) (c) $a = -2b$

Chapter 2

(Page 40)

A decreasing sequence of fractions starting at 1 or a decreasing sequence of percentages starting at 100%

e.g. 100%, 95%, 80%, 55%, 50%, 20%, 5%

(Page 41)

Start at 2 and add 3 each time.

Exercise 2A (Page 45)

1 (i) 6, 11, 16, 21, 26
Increasing by 5 for each term

(ii) −3, −9, −15, −21, −27
Decreasing by 6 for each term

(iii) 8, 16, 32, 64, 128
Doubling for each term

(iv) 8, 12, 8, 12, 8
Oscillating

(v) 2, 5, 11, 23, 47
Increasing

(vi) $5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, 1$
Decreasing, converging to zero

2 (i) 21, 25, 29, 33
(ii) $u_1 = 1,\ u_{r+1} = u_r + 4$
(iii) $u_r = 4r - 3$

3 (i) (a) $0, -2, -4, -6$
(b) $u_1 = 10, u_{r+1} = u_r - 2$
(c) $u_r = 12 - 2r$
(d) -28
(ii) (a) 32, 64, 128, 256
(b) $u_1 = 1, u_{r+1} = 2u_r$
(c) $u_r = 2^{r-1}$
(d) 524 288
(iii) (a) 31 250, 156 250, 781 250, 3 906 250
(b) $u_1 = 50, u_{r+1} = 5u_r$
(c) $u_r = 10 \times 5^r$
(d) 9.54×10^{14}

4 (i) 25 (ii) -150
(iii) 363 (iv) -7.5

5 (i) $\sum_{r=1}^{7}(56 - 6r)$ (ii) 224

6 2500

7 (i) $-5, 5, -5, 5, -5, 5$
Oscillating
(ii) (a) 0 (b) -5
(iii) $-\frac{5}{2} + \frac{5}{2}(-1)^n$

8 (i) 0, 100, 2, 102, 4, 104
Even terms start from 100 and increase by 2, odd terms start from 0 and increase by 2.
(ii) 201
(iii) 102

9 749 cm

10 $\frac{1}{2}n(n^3 + 1)$

11 10, 5, 16, 8, 4 (This will reach 1 at c_7 and then repeat the cycle 4, 2, 1)

Exercise 2B (Page 49)

1 (i) 1, 3, 5 (ii) n^2

2 (i) 4, 14, 30 (ii) $n(n + 1)^2$

3 (i) 2, 12, 36 (ii) $\frac{1}{12}n(n + 1)(3n + 1)(n + 2)$

4 n^4

5 $\frac{1}{3}n(n + 1)(n + 2)$

6 $\frac{1}{4}n(n + 1)(n + 2)(n + 3)$

7 $\frac{1}{2}n(3n + 1)$

8 $n^2(4n + 1)(5n + 2)$

9 (ii) 7 layers, 125 left over

10 (i) \$227.50
(ii) $\frac{1}{24}n(35(n + 1) + 30I)$

11 $S_1 = 3 \quad S_2 = 10$
$S_3 = 21 \quad S_4 = 36$

$u_r = 4r - 1$

$\sum_{r=n+1}^{2n} u_r = 6n^2 + n$

? (Page 51)

As n becomes very large, the top and bottom of $\frac{n}{n+1}$ are very close, so the sum becomes very close to 1 (it converges to 1).

? (Page 53)

As n becomes very large, the expression $\frac{n(3n + 7)}{2(n + 1)(n + 2)}$ becomes close to $\frac{3n^2}{2n^2}$ (since terms in n^2 are much bigger than terms in n). So the sum becomes very close to $\frac{3}{2}$ $\left(\text{it converges to } \frac{3}{2}\right)$.

Exercise 2C (Page 54)

1 (ii) $(1 - 0) + (4 - 1) + (9 - 4) + \ \dots\ +$
$[(n - 2)^2 - (n - 3)^2] +$
$[(n - 1)^2 - (n - 2)^2] + [n^2 - (n - 1)^2]$
(iii) n^2

2 (i) First term: $r = 1$, last term: $r = 10$
(iii) $\frac{20}{21}$

3 (ii) $n(n^2 + 4n + 5)$ (iv) 99

4 (ii) $\dfrac{n(n+2)}{(n+1)^2}$

5 (ii) $\dfrac{n(3n+5)}{4(n+1)(n+2)}$

(iii) 0.7401, 0.7490, 0.7499. The sum looks as if it is approaching 0.75 as n becomes large.

6 (ii) $\dfrac{13}{120}$

7 (ii) $\dfrac{n(n+3)}{4(n+1)(n+2)}$

(iii) 0.249 95…, 0.249 9995… The sum looks as if it is approaching 0.25 as n becomes large.

8 (ii) $8n^3 + 12n^2 + 6n$

9 (ii) $16n^4 + 32n^3 + 24n^2 + 8n$

10 (ii) $A = \dfrac{1}{2}, B = -\dfrac{1}{2}$

(iii) $\dfrac{(3n+2)(n-1)}{4n(n+1)}$

(iv) As $n \to \infty$, the sum $\to \dfrac{3}{4}$

11 $\displaystyle\sum_{k=13}^{n} u_k = \frac{1}{5} - \frac{1}{\sqrt{2n+1}}$

$\displaystyle\sum_{k=13}^{\infty} u_k = \frac{1}{5}$

12 (i) 4.820

(ii) 55

13 $\dfrac{1}{2}n^2(n+1)$

? (Page 56)

If she was 121 last year then it would be fine, but we don't know if this is true. If she were able to provide any evidence of her age at a particular point then we could work from there, but we need a starting point.

Activity 2.1 (Page 56)

$$\frac{1}{1\times 2} = \frac{1}{2}$$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} = \frac{2}{3}$$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} = \frac{3}{4}$$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} = \frac{4}{5}$$

Activity 2.2 (Page 60)

(i) Assume true for $n = k$, so

$$2 + 4 + 6 + \ldots + 2k = \left(k + \frac{1}{2}\right)^2.$$

For $n = k + 1$,

$$\begin{aligned} 2 + 4 + 6 + \ldots &+ \\ 2k + 2(k+1) &= \left(k + \frac{1}{2}\right)^2 + 2(k+1) \\ &= k^2 + k + \frac{1}{4} + 2k + 2 \\ &= k^2 + 3k + \frac{9}{4} \\ &= \left(k + \frac{3}{2}\right)^2 \\ &= \left(k + 1 + \frac{1}{2}\right)^2 \end{aligned}$$

It is not true for $n = 1$.

(ii) It breaks down at the inductive step.

Exercise 2E (Page 65)

5 (ii) **M** is a shear, x-axis fixed, (0, 1) maps to (1, 1).
$\mathbf{M}^n$ is a shear, x-axis fixed, (0, 1) maps to (n, 1).

6 (i) $u_2 = \dfrac{1}{2}, u_3 = \dfrac{1}{3}, u_4 = \dfrac{1}{4}$

(ii) $u_n = \dfrac{1}{n}$

7 (i) $\mathbf{A}^2 = \begin{pmatrix} -3 & -8 \\ 2 & 5 \end{pmatrix}, \mathbf{A}^3 = \begin{pmatrix} -5 & -12 \\ 3 & 7 \end{pmatrix}$

8 (i) $\mathbf{M}^2 = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix},$

$\mathbf{M}^3 = \begin{pmatrix} -7 & 14 \\ 21 & 7 \end{pmatrix}, \mathbf{M}^4 = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}$

(ii) $\mathbf{M}^{2m} = 7^m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{M}^{2m+1} = 7^m\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$

9 (i) 3, 5, 17, 257, 65 537

10 $2! - S_1 = 1,\ 3! - S_2 = 1,\ 4! - S_3 = 1,\ 5! - S_4 = 1.$

$S_n = (n+1)! - 1$

11 $\dfrac{2}{1+x}$

Chapter 3

(Page 69)

$4x^3 + x^2 - 4x - 1 = 0$

Looking at the graph you may suspect that $x = 1$ is a root. Setting $x = 1$ verifies this. The factor theorem tells you that $(x - 1)$ must be a factor, so factorise the cubic $(x-1)(4x^2 + 5x + 1) = 0$. Now factorise the remaining quadratic factor: $(x-1)(4x+1)(x+1) = 0$, so the roots are $x = 1, -\frac{1}{4}, -1$.

$4x^3 + x^2 + 4x + 1 = 0$

This does not have such an obvious starting point, but the graph suggests only one real root.

Comparing with previous example, you may spot that $x = -\frac{1}{4}$ might work, so you can factorise giving $(4x+1)(x^2+1) = 0$. From this you can see that the other roots must be complex. $x^2 = -1$, so the three roots are $x = -\frac{1}{4}, \pm\mathrm{i}$.

Activity 3.1 (Page 71)

Equation	Two roots	Sum of roots	Product of roots
(i) $z^2 - 3z + 2 = 0$	1, 2	3	2
(ii) $z^2 + z - 6 = 0$	2, –3	–1	–6
(iii) $z^2 - 6z + 8 = 0$	2, 4	6	8
(iv) $z^2 - 3z - 10 = 0$	–2, 5	3	–10
(v) $2z^2 - 3z + 1 = 0$	$\frac{1}{2}$, 1	$\frac{3}{2}$	$\frac{1}{2}$
(vi) $z^2 - 4z + 5 = 0$	$2 \pm \mathrm{i}$	4	5

(Page 71)

If the equation is $ax^2 + bx + c = 0$, the sum appears to be $-\frac{b}{a}$ and the product appears to be $\frac{c}{a}$.

(Page 72)

You get back to the original quadratic equation.

Activity 3.3 (Page 75)

(i) $\dfrac{-3 \pm i\sqrt{31}}{4}, \dfrac{-3 \pm i\sqrt{31}}{2}$

(ii) $\dfrac{2 \pm \sqrt{7}}{3}, \dfrac{5 \pm \sqrt{7}}{3}$

Exercise 3A (Page 75)

1 (i) $\alpha + \beta = -\frac{7}{2}, \quad \alpha\beta = 3$

(ii) $\alpha + \beta = \frac{1}{5}, \quad \alpha\beta = -\frac{1}{5}$

(iii) $\alpha + \beta = 0, \quad \alpha\beta = \frac{2}{7}$

(iv) $\alpha + \beta = -\frac{24}{5}, \quad \alpha\beta = 0$

(v) $\alpha + \beta = -11, \quad \alpha\beta = -4$

(vi) $\alpha + \beta = -\frac{8}{3}, \quad \alpha\beta = -2$

2 (i) $z^2 - 10z + 21 = 0$

(ii) $z^2 - 3z - 4 = 0$

(iii) $2z^2 + 19z + 45 = 0$

(iv) $z^2 - 5z = 0$

(v) $z^2 - 6z + 9 = 0$

(vi) $z^2 - 6z + 13 = 0$

3 (i) $2z^2 + 15z - 81 = 0$

(ii) $2z^2 - 5z - 9 = 0$

(iii) $2z^2 + 13z + 9 = 0$

(iv) $z^2 - 7z - 12 = 0$

4 $z^2 - 20z + 4 = 0$

5 (i) $(\alpha + \beta)^2 - 2\alpha\beta$

(ii) $\dfrac{\alpha + \beta}{\alpha\beta}$

(iii) $\dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

(iv) $\alpha\beta(\alpha + \beta)$

(v) $(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

6 (i) Roots are real, distinct and negative (since $\alpha\beta > 0 \Rightarrow$ same signs and $\alpha + \beta < 0 \Rightarrow$ both <0)

(ii) $\alpha = -\beta$

(iii) One of the roots is zeros and the other is $-\frac{b}{a}$.

(iv) The roots are of opposite signs.

10 (i) $az^2 + bkz + ck^2 = 0$

(ii) $az^2 + (b - 2ka)z + (k^2a - kb + c) = 0$

11 (ii) $z^2 - (5 + 2i)z + (9 + 7i) = 0$

Exercise 3B (Page 80)

1 (i) $-\frac{3}{2}$

(ii) $-\frac{1}{2}$

(iii) $-\frac{7}{2}$

2 (i) $z^3 - 7z^2 + 14z - 8 = 0$

(ii) $z^3 - 3z^2 - 4z + 12 = 0$

(iii) $2z^3 + 7z^2 + 6z = 0$

(iv) $2z^3 - 13z^2 + 28z - 20 = 0$

(v) $z^3 - 19z - 30 = 0$

(vi) $z^3 - 5z^2 + 9z - 5 = 0$

3 (i) $z = 2, 5, 8$

(ii) $z = -\frac{2}{3}, \frac{2}{3}, 2$

(iii) $z = 2 - 2\sqrt{3}, 2, 2 + 2\sqrt{3}$

(iv) $z = \frac{2}{3}, \frac{7}{6}, \frac{5}{3}$

4 (i) $z = w - 3$

(ii) $(w - 3)^3 + (w - 3)^2 + 2(w - 3) - 3 = 0$

(iii) $w^3 - 8w^2 + 23w - 27 = 0$

(iv) $\alpha + 3, \beta + 3, \gamma + 3$

5 (i) $w^3 - 4w^2 + 4w - 24 = 0$

(ii) $z^3 - 2z^2 - 11z - 9 = 0$

6 (i) $2w^3 - 16w^2 + 37w - 27 = 0$

(ii) $2w^3 + 24w^2 + 45w + 37 = 0$

(iii) $z^3 - 28z^2 + z - 1 = 0$

7 The roots are $\frac{3}{2}, 2, \frac{5}{2}$ $\quad k = \frac{47}{2}$

8 $z = \frac{1}{4}, \frac{1}{2}, -\frac{3}{4}$

9 $\alpha = -1, p = 7, q = 8$ or $\alpha = p = q = 0$

12 Roots are $-p$ and $\pm\sqrt{-q}$ (note $\pm\sqrt{-q}$ is not necessarily imaginary, since q is not necessarily >0)

13 (i) $p = -8\left(\alpha + \frac{1}{2\alpha} + \beta\right)$

$q = 8\left(\frac{1}{2} + \alpha\beta + \frac{\beta}{2\alpha}\right)$

$r = -4\beta$

(iii) $r = 9; x = 1, \frac{1}{2}, -\frac{9}{4}$

$r = -6; x = -2, -\frac{1}{4}, \frac{3}{2}$

14 $z = \frac{3}{7}, \frac{7}{3}, -2$

15 $ac^3 = b^3d$

$z = \frac{1}{2}, \frac{3}{2}, \frac{9}{2}$

16 $p = -15,\ q = 71\ ; \alpha = 3,\ \ r = -105$

? (Page 82)

See derivation of formulae on page 83.

$\sum\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$

$\sum\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

$\sum\alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a}$

$\alpha\beta\gamma\delta = \frac{e}{a}$

Exercise 3C (Page 84)

1 (i) $-\frac{3}{2}$

(ii) 3

(iii) $\frac{5}{2}$

(iv) 2

2 (i) $z^4 - 6z^3 + 7z^2 + 6z - 8 = 0$

(ii) $4z^4 + 20z^3 + z^2 - 60z = 0$

(iii) $4z^4 + 12z^3 - 27z^2 - 54z + 81 = 0$

(iv) $z^4 - 5z^2 + 10z - 6 = 0$

3 (i) $z^4 + 4z^3 - 6z^2 + 8z + 48 = 0$

(ii) $2z^4 + 12z^3 + 21z^2 + 13z + 8 = 0$

4 (i) Let $w = x + 1$ then $x = w - 1$

new quartic: $x^4 - 6x^2 + 9$

(ii) Solutions to new quartic are $x = \pm\sqrt{3}$ (each one repeated), solutions to original quartic are therefore: $\alpha = \beta = \sqrt{3} - 1$ and $\gamma = \delta = -\sqrt{3} - 1$.

5 (i) $\alpha = -1,\ \beta = \sqrt{3}$

(ii) $p = 4$ and $q = -9$

(iii) Use substitution $y = x - 3\alpha$ (i.e. $y = x + 3$ then $x = y - 3$) and

$$y^3 - 8y^2 + 18y - 12 = 0$$

6 (i) $\alpha + \beta + \gamma + \delta + \varepsilon = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \alpha\varepsilon + \beta\gamma + \beta\delta + \beta\varepsilon + \gamma\delta + \gamma\varepsilon + \delta\varepsilon = \frac{c}{a}$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\beta\varepsilon + \alpha\gamma\delta + \alpha\gamma\varepsilon + \alpha\delta\varepsilon + \beta\gamma\delta + \beta\gamma\varepsilon + \beta\delta\varepsilon + \gamma\delta\varepsilon = -\frac{d}{a}$

$\alpha\beta\gamma\delta + \beta\gamma\delta\varepsilon + \gamma\delta\varepsilon\alpha + \delta\varepsilon\alpha\beta + \varepsilon\alpha\beta\gamma = \frac{e}{a}$

$\alpha\beta\gamma\delta\varepsilon = -\frac{f}{a}$

$\sum\alpha = -\frac{b}{a}$

$\sum\alpha\beta = \frac{c}{a}$

$\sum\alpha\beta\gamma = -\frac{d}{a}$

$\sum\alpha\beta\gamma\delta = \frac{e}{a}$

$\sum\alpha\beta\gamma\delta\varepsilon = \alpha\beta\gamma\delta\varepsilon = -\frac{f}{a}$

7 (i) -4

(ii) 12

(iii) 4

(iv) 12

$y^4 - 4y^2 + 4 = 0;$

$y = \pm\sqrt{2}$ (twice)

$x = \pm\sqrt{2} - 1$ (twice)

8 (i) $S_2 = 6$, $S_4 = 26$

(ii) $S_3 = -15,\ S_5 = -75$

-15

Chapter 4

(Page 88)

The population of rabbits fluctuates, but eventually approaches a stable number.

(Page 90)

(i) $x = -2$

(ii) $x = 1$ and $x = -2$

(iii) $x = \frac{1}{2}$

(Page 90)

(i) $y = 0$

(ii) $y = 1$

(iii) $y = -2$

(Page 93)

(i) positive, positive, positive

(ii) negative, negative, negative

(iii) positive

(iv) negative

(Page 93)

$$y = \frac{(x+2)}{(x-2)(x+1)} = \frac{(x+2)}{(x^2 - x - 2)}$$

The quotient rule for $y = \frac{u}{v}$ is $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Using this with $u = x + 2$ and $v = x^2 - x - 2$ gives

$$\frac{dy}{dx} = \frac{(x^2 - x - 2) \times 1 - (x+2)(2x-1)}{(x^2 - x - 2)^2}$$

$$= \frac{-x^2 - 4x}{(x^2 - x - 2)^2}$$

$$= \frac{-x(x+4)}{(x^2 - x - 2)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow -x(x+4) = 0 \Rightarrow x = 0,\ x = -4$$

When $x = 0$ then $y = \frac{(0+2)}{(0-2)(0+1)} = -1$

When $x = -4$ then $y = \frac{(-4+2)}{(-4-2)(-4+1)} = -\frac{1}{9}$

Exercise 4A (Page 96)

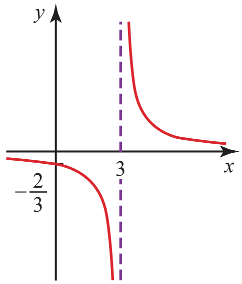
1 Step 1: $\left(0, -\frac{2}{3}\right)$

Step 2: $x = 3$

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from below as $x \to -\infty$

Step 4:



Step 5: $y \neq 0$

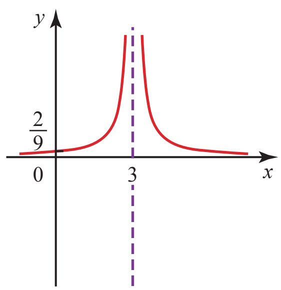
2 Step 1: $\left(0, -\frac{2}{9}\right)$

Step 2: $x = 3$

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from above as $x \to -\infty$

Step 4:



Step 5: $y > 0$

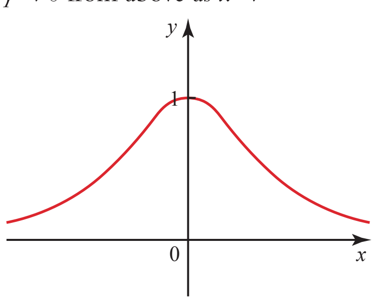
3 Step 1: (0, 1)

Step 2: none

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from above as $x \to -\infty$

Step 4:



Step 5: $0 < y \leqslant 0$

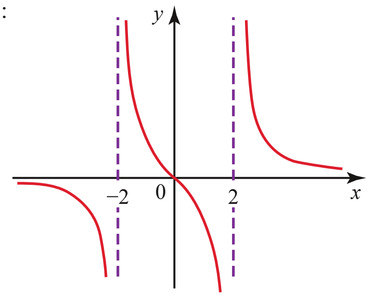
4 Step 1: (0, 0)

Step 2: $x = 2, x = -2$

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from below as $x \to -\infty$

Step 4:



Step 5: $y \in \mathbb{R}$

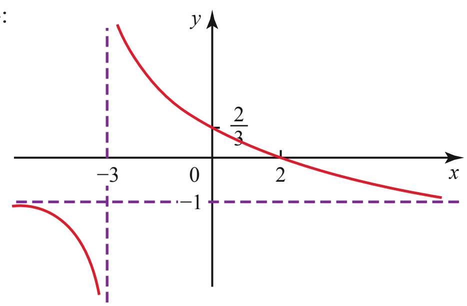
5 Step 1: (2, 0), $\left(0, \frac{2}{3}\right)$

Step 2: $x = -3$

Step 3: $y \to -1$ from above as $x \to \infty$

$y \to -1$ from below as $x \to -\infty$

Step 4:



Step 5: $y \neq 1$

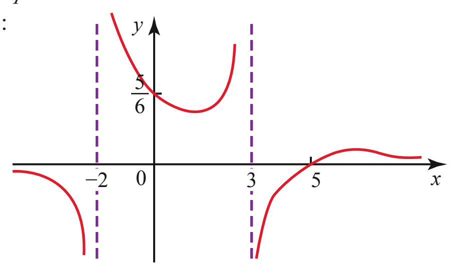
6 Step 1: (5, 0), $\left(0, \frac{5}{6}\right)$

Step 2: $x = -2, x = 3$

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from below as $x \to -\infty$

Step 4:



Step 5: $y \leqslant \frac{9 - 2\sqrt{14}}{25}, \ y \geqslant \frac{9 + 2\sqrt{14}}{25}$

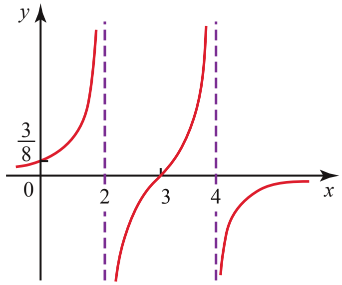
7 Step 1: (3, 0), $\left(0, \frac{3}{8}\right)$

Step 2: $x = 2, x = 4$

Step 3: $y \to 0$ from below as $x \to \infty$

$y \to 0$ from above as $x \to -\infty$

Step 4:



Step 5: $y \in \mathbb{R}$

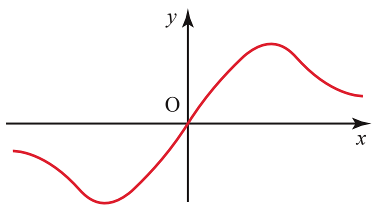
8 Step 1: (0, 0)

Step 2: none

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from below as $x \to -\infty$

Step 4:



Step 5: $-\frac{1}{2\sqrt{3}} \leqslant y \leqslant \frac{1}{2\sqrt{3}}$

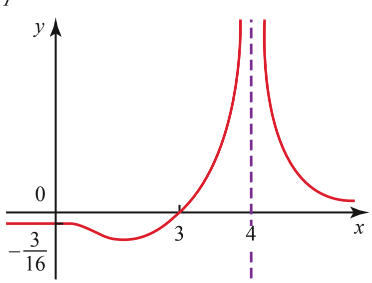
9 Step 1: (3, 0), $\left(0, -\frac{3}{16}\right)$

Step 2: $x = 4$

Step 3: $y \to 0$ from above as $x \to \infty$

$y \to 0$ from below as $x \to -\infty$

Step 4:



Step 5: $y \geqslant -\frac{1}{4}$

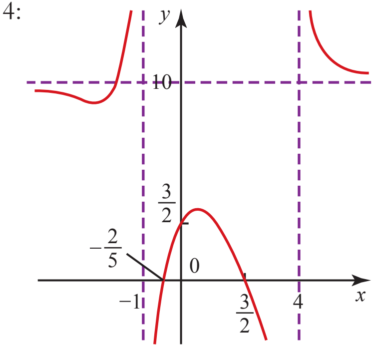
10 Step 1: $\left(\frac{3}{2}, 0\right)$, $\left(-\frac{2}{5}, 0\right)$, $\left(0, \frac{3}{2}\right)$

Step 2: $x = -1$, $x = 4$

Step 3: $y \to 10$ from above as $x \to \infty$

$y \to 10$ from below as $x \to -\infty$

Step 4:



Step 5: $y \leqslant \frac{25 - 2\sqrt{66}}{5}$, $y \geqslant \frac{25 + 2\sqrt{66}}{5}$

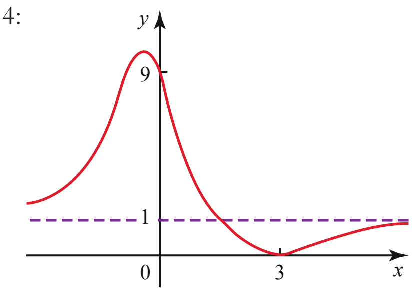
11 Step 1: (3, 0) (repeated), (0, 9)

Step 2: none

Step 3: $y \to 1$ from below as $x \to \infty$

$y \to 1$ from above as $x \to -\infty$

Step 4:



Step 5: $0 \leqslant y \leqslant 10$

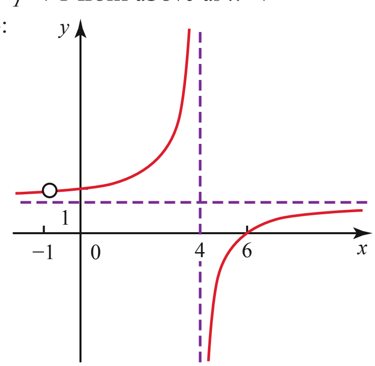
12 Step 1: (6, 0), $\left(0, \frac{3}{2}\right)$

Step 2: $x = 4$

Step 3: $y \to 1$ from below as $x \to \infty$

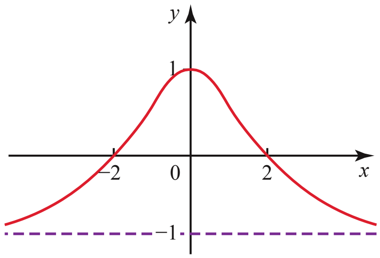
$y \to 1$ from above as $x \to -\infty$

Step 4:



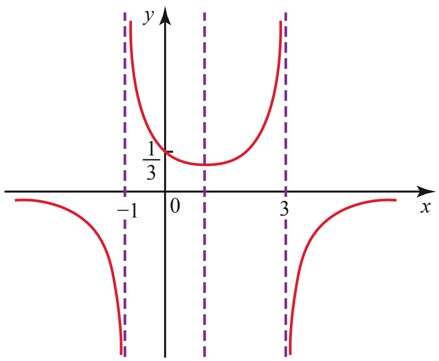
Step 5: $y \neq 1$, $y \neq \frac{7}{5}$

13 (i)



(ii) $k \leqslant -1$ or $k > 1$

14 (i)



(ii) $x = 1$, minimum point $= \left(1, \frac{1}{4}\right)$

(iii) (a) $k < 0,\ k > \frac{1}{4}$

(b) $k = \frac{1}{4}$

(c) $0 \leqslant k < \frac{1}{4}$

15 (i) $x \leqslant -2$ or $x > 1$

(ii) $-5 \leqslant x < 2$

(iii) $-1 < x \leqslant 2$ or $3 < x \leqslant 7$

(iv) $\frac{1}{2} < x \leqslant \frac{5}{3}$

(v) $-3 < x \leqslant \frac{5}{3}$

(vi) $x < -6$ or $-2 \leqslant x < \frac{2}{3}$

Exercise 4B (Page 100)

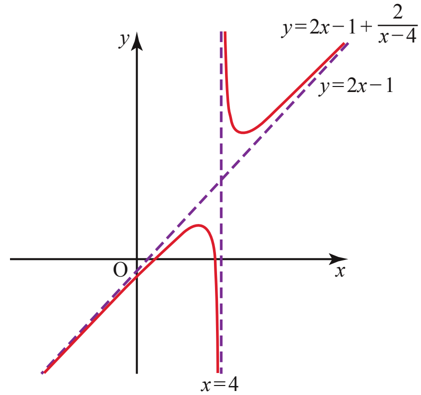
1 (i) (a) $\left(\frac{1}{4}(9 - \sqrt{33}),\ 0\right)$, $\left(\frac{1}{4}(9 + \sqrt{33}),\ 0\right)$,

$(0,\ -\frac{3}{2})$

(b) $y = 2x - 1;\ x = 4$

(c) $(3, 3); (5, 11)$

(d)



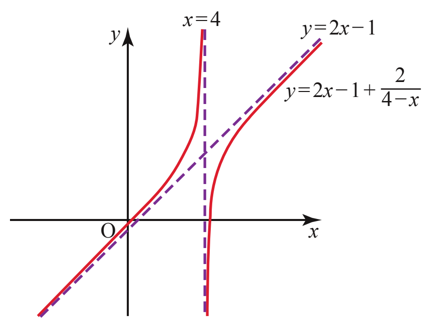
(ii) (a) $\left(\frac{1}{4}(9 - \sqrt{65}),\ 0\right)$, $\left(\frac{1}{4}(9 + \sqrt{65}),\ 0\right)$,

$(0,\ -\frac{1}{2})$

(b) $y = 2x - 1;\ x = 4$

(c) No turning points

(d)



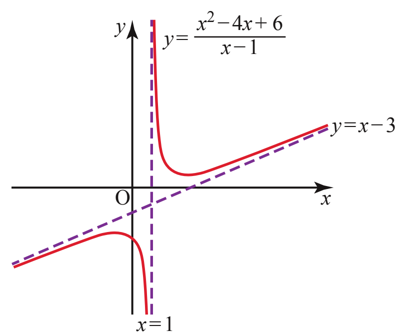
2 (i) (a) $(0, -6)$

(b) $y = 3 - x;\ x = 1$

(c) $((1 - \sqrt{3},\ -2(1 + \sqrt{3}));$

$((1 + \sqrt{3},\ 2(\sqrt{3} - 1))$

(d)



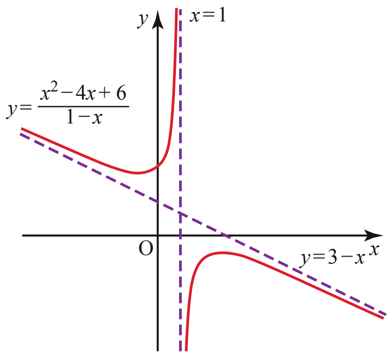
Answers

(ii) (a) $(0, 6)$

(b) $y = 3 - x;\ x = 1$

(c) $((1 - \sqrt{3},\ 2(1 + \sqrt{3}));$
$((1 + \sqrt{3},\ 2(1 - \sqrt{3}))$

(d)

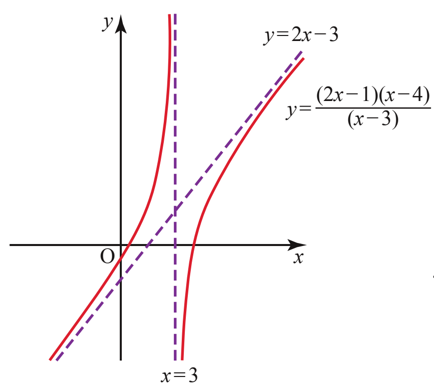


3 (i) (a) $\left(\frac{1}{2}, 0\right)$, $(4, 0)$, $\left(0, -\frac{4}{3}\right)$

(b) $y = 2x - 3;\ x = 3$

(c) No turning points

(d)

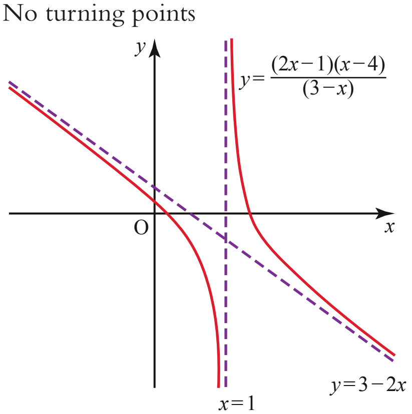


(ii) (a) $\left(\frac{1}{2}, 0\right), (4, 0), \left(0, \frac{4}{3}\right)$

(b) $y = 3 - 2x;\ x = 3$

(c) No turning points

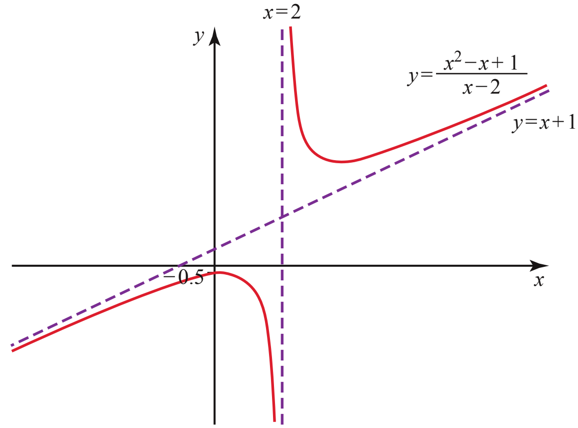
(d)



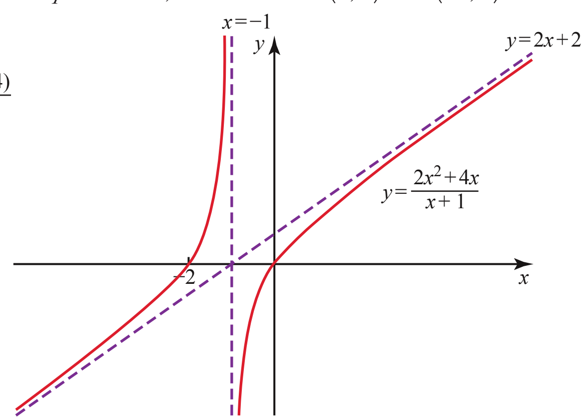
4 Vertical asymptote: $x = 1$; oblique asymptote: $y = 2x + 3$

5 Vertical asymptote: $x = 2$; oblique asymptote: $y = x + p + 2$

$p > -\frac{5}{2}$

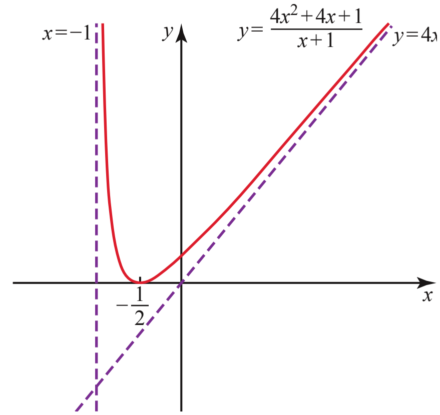


6 No stationary points when $k > 2$
vertical asymptote: $x = -1$; oblique asymptote: $y = 2x + 2$; crosses axes at $(0, 0)$ and $(-2, 0)$

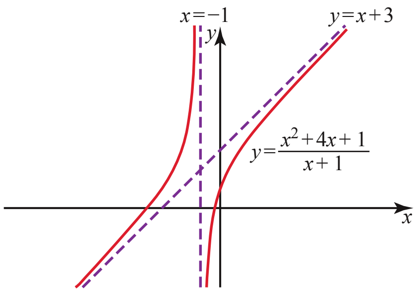


7 (i) Vertical asymptote: $x = -1$; oblique asymptote: $y = px + 4 - p$

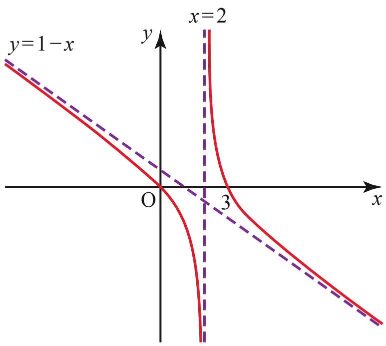
(ii) $p = 4$



(iii) Intersections at $(-2 \pm \sqrt{3}, 0)$



8 Vertical asymptote: $x = 2$; oblique asymptote: $y = \lambda x + 1$



❓ (Page 104)

Using the chain rule:

Let $y = \dfrac{1}{\mathrm{f}(x)}$ and $u = \mathrm{f}(x)$

Then $y = \dfrac{1}{u} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}u} = -\dfrac{1}{u^2}$

and $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{f}'(x)$

$$\text{So } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$= -\frac{1}{u^2} \times \mathrm{f}'(x)$$

$$= -\frac{\mathrm{f}'(x)}{[\mathrm{f}(x)]^2}$$

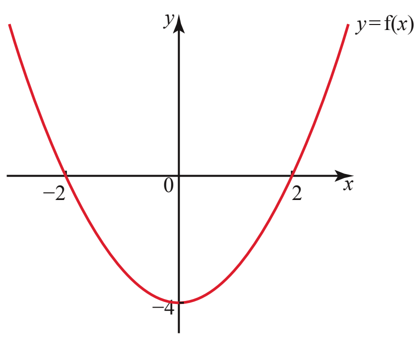
❓ (Page 107)

The graphs are the same shape, but $y = \dfrac{1}{\mathrm{f}(x)}$ is undefined when $x = 1$ since $\mathrm{f}(x)$ is undefined at that point.

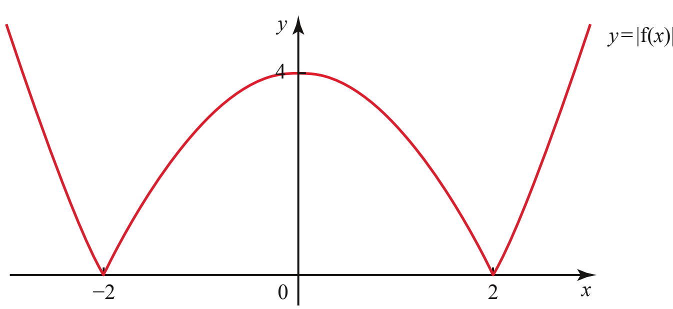
$y = g(x)$ has a root at $x = 1$.

Exercise 4C (Page 110)

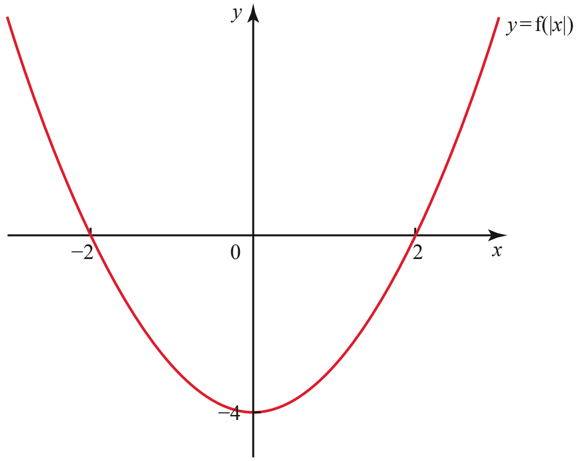
1 **(i)**



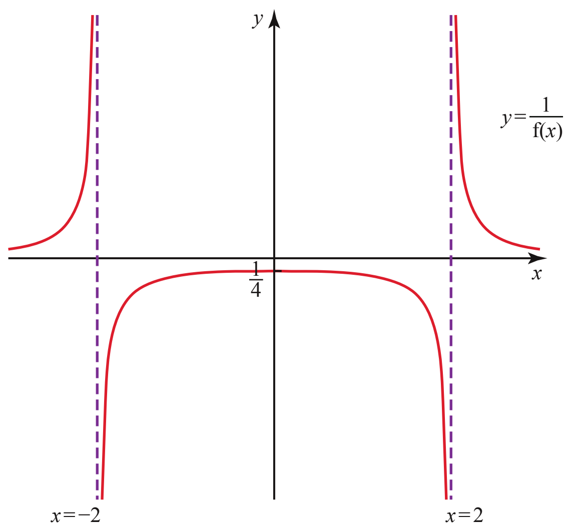
(ii)



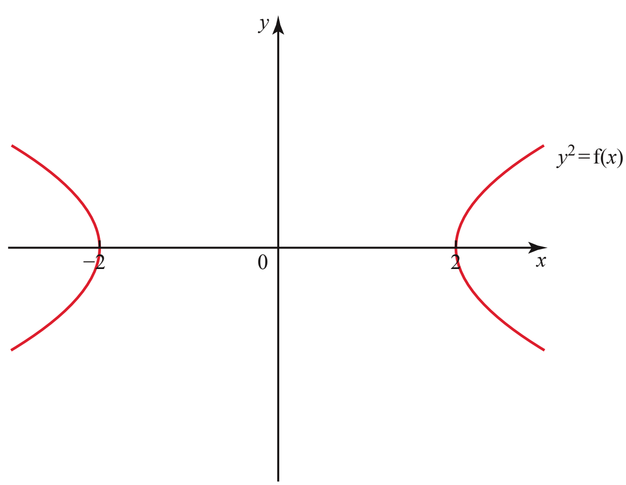
(iii)



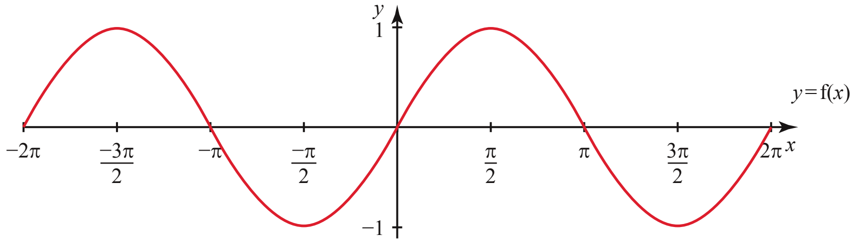
(iv)



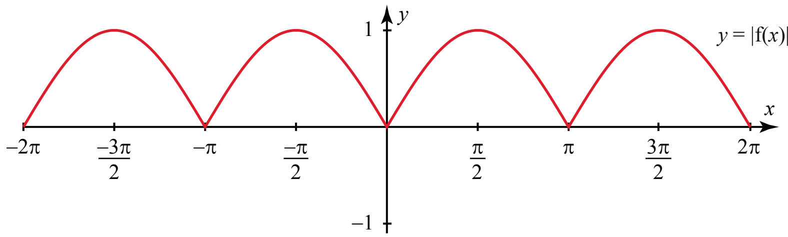
(v)



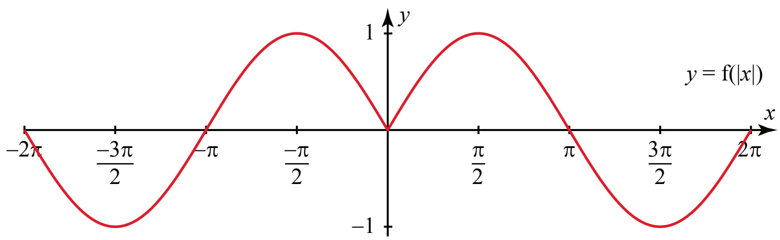
2 (i)


y
1
−1
x
y = f(x)
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π

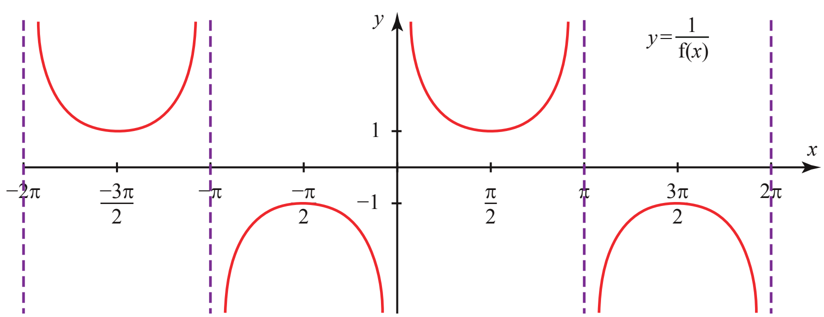
(ii)


y
1
−1
x
y = |f(x)|
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π

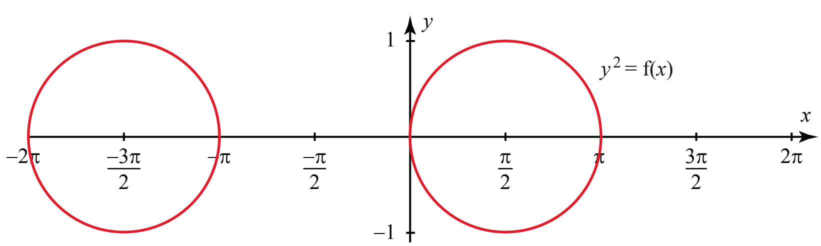
(iii)


y
1
−1
x
y = f(|x|)
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π

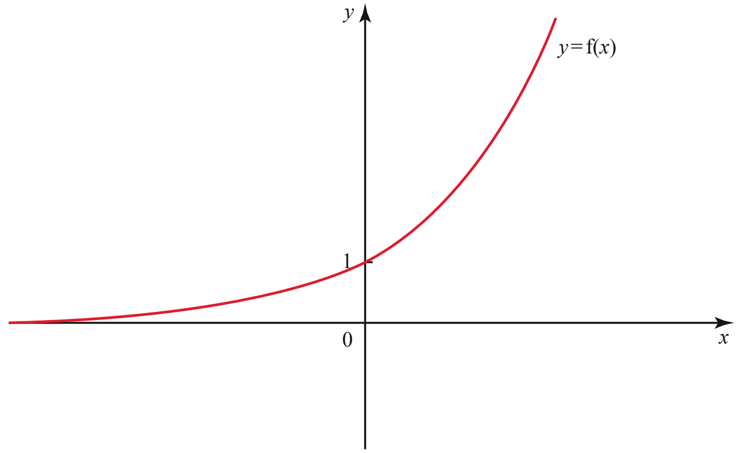
(iv)


y
1
−1
x
y = 1/f(x)
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π

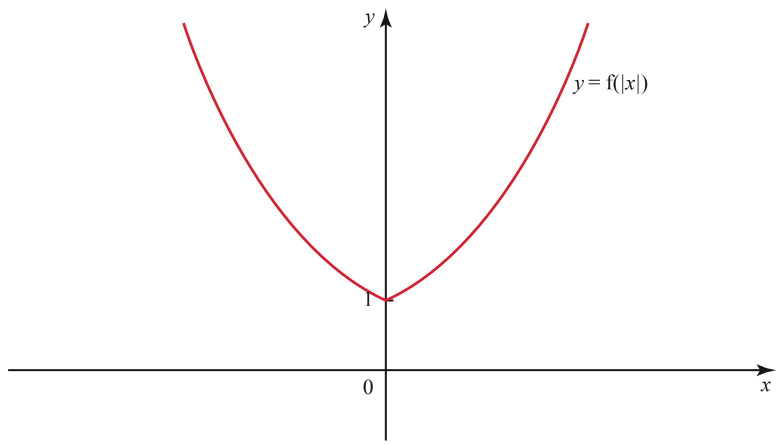
(v)


y
1
−1
x
y² = f(x)
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π

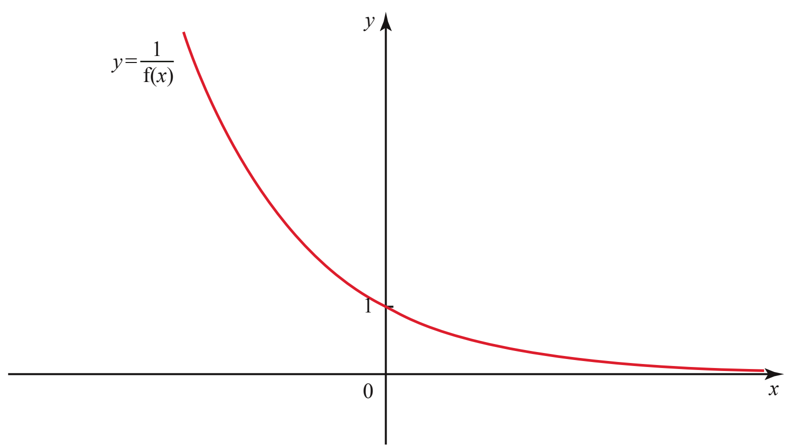
3 (i), (ii)



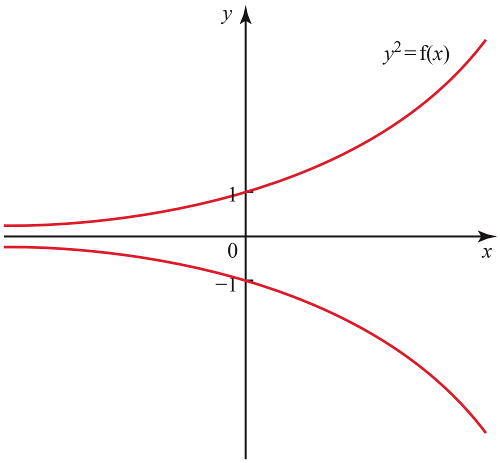
(iii)



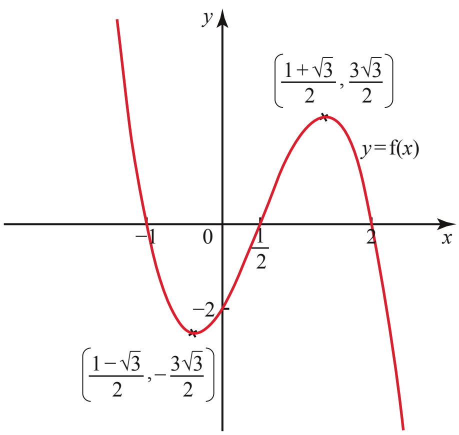
(iv)



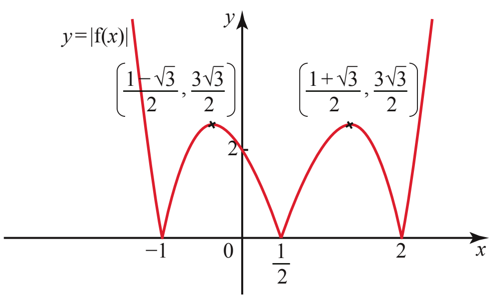
(v)



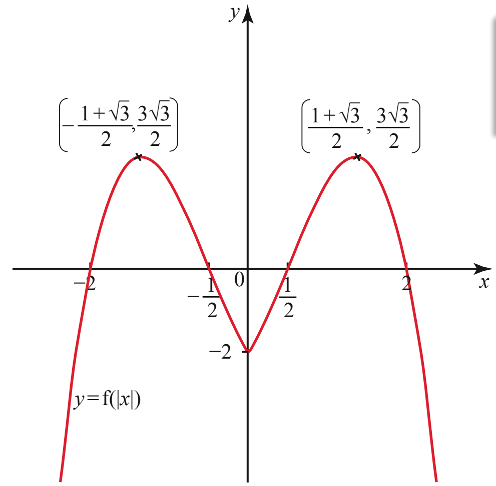
4 (i)



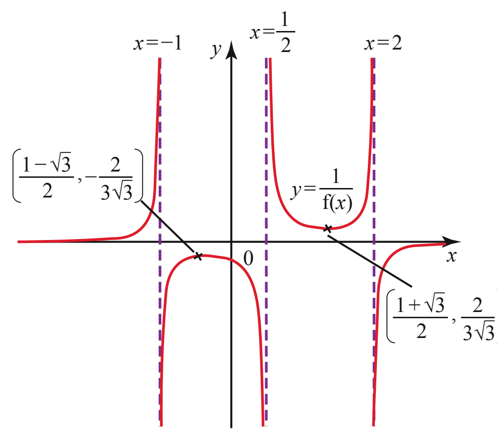
(ii)



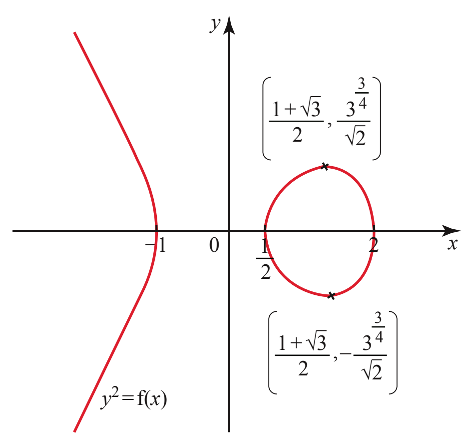
(iii)



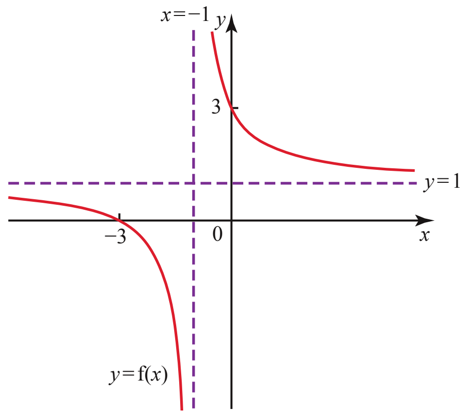
(iv)



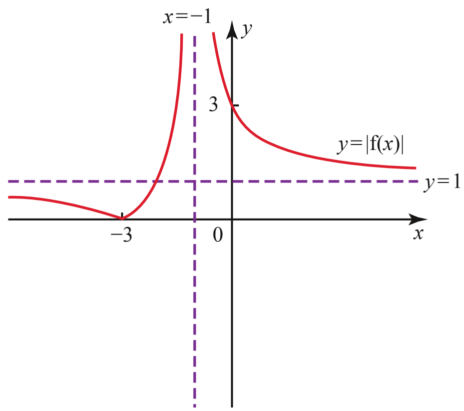
(v)



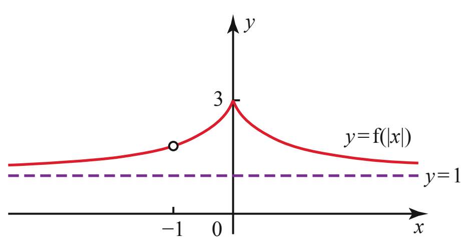
5 (i)



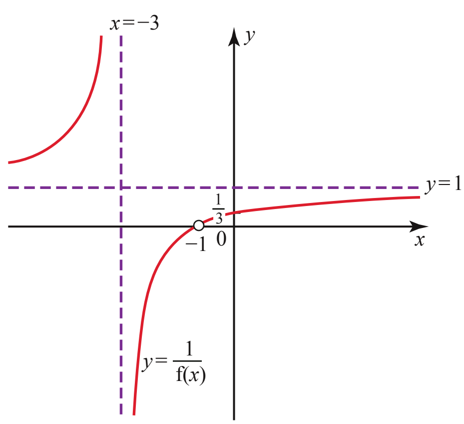
(ii)



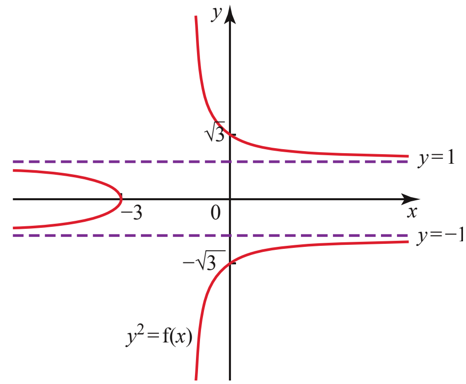
(iii)



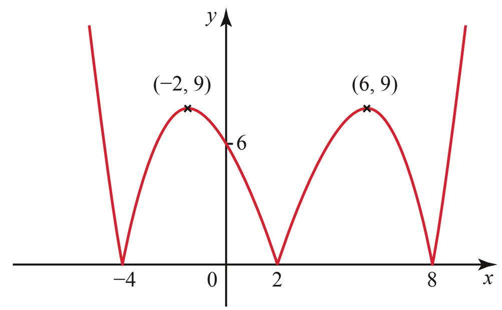
(iv)



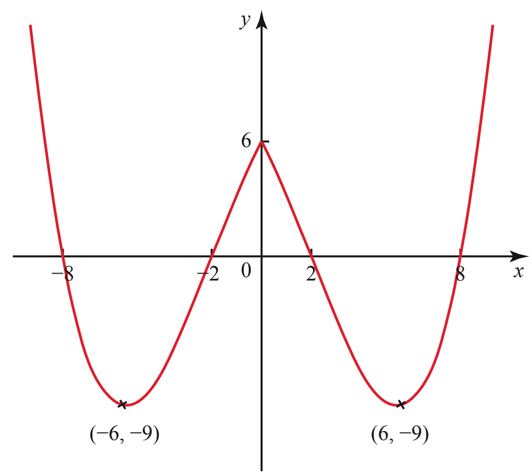
(v)



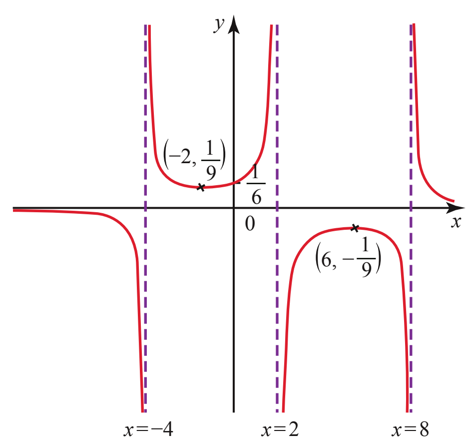
6 (i)



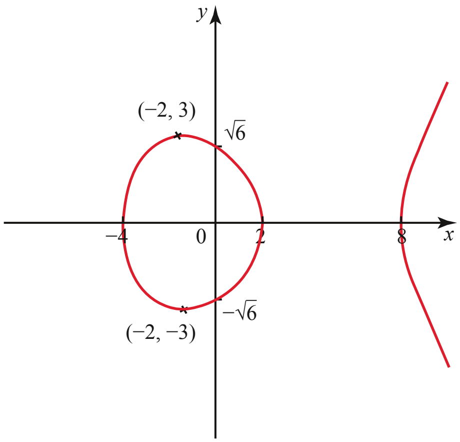
(ii)



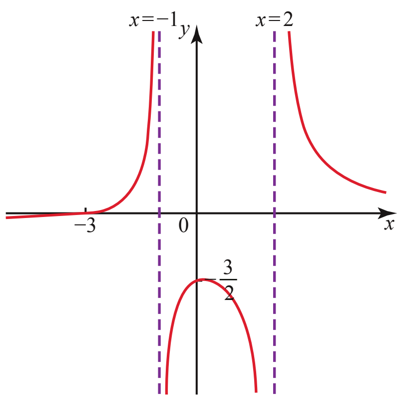
(iii)



(iv)

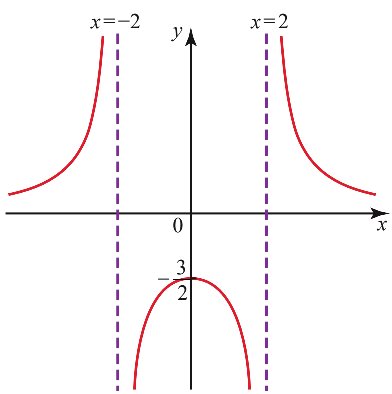


7 (i) (a)



(b) $x \leqslant -3$ or $-1 < x < 2$

(ii) (a)



(b) $-2 < x < 2$

8 (i) (a) $y = 0$

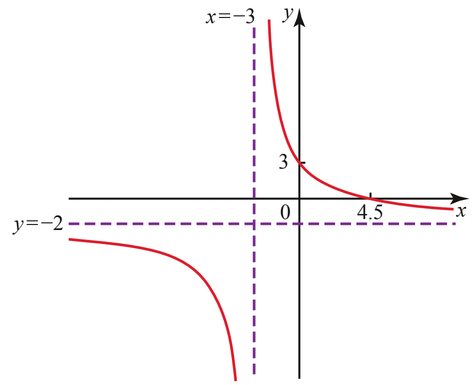
(b) $-2 < y < 2$

(ii) (a) $y = 0$

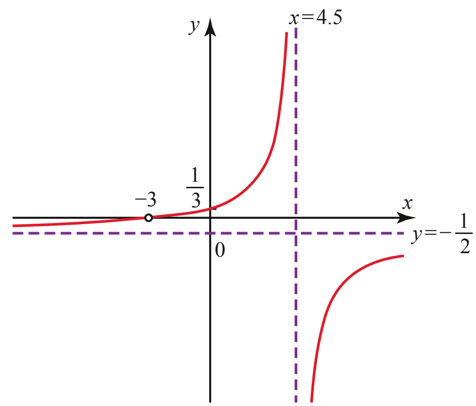
(b) maximum $\sqrt{2}$ and minimum $-\sqrt{2}$

(c) $x \geqslant 0$

9 (i)



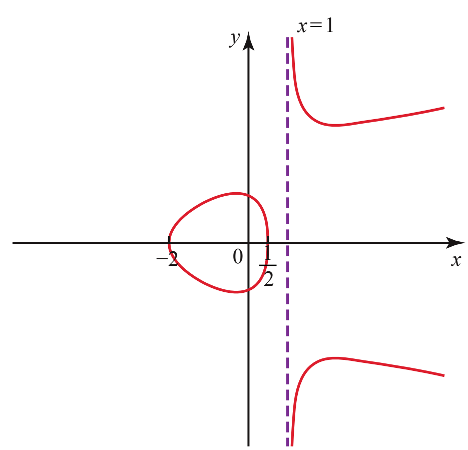
(ii) $x = -3$ and $x = 4.5$



10 (i) $y = \dfrac{2x^2 + 3x - 2}{x - 1}$

(ii) $y \leq 7 - 2\sqrt{6}$ or $y \geqslant 7 + 2\sqrt{6}$

(iii)



Answers

Chapter 5

(Page 115)

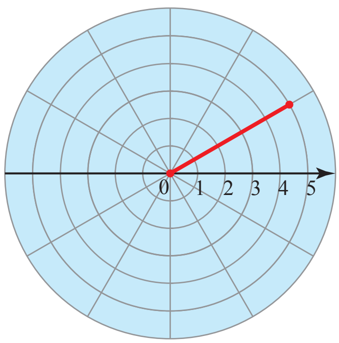
See text that follows on page 115.

(Page 116)

There are an infinite number of pairs of polar co-ordinates (r, θ) to define a given point P.

Activity 5.1 (Page 116)

All the points are in the position shown below:



For example, $\left(6, -\frac{5\pi}{4}\right), \left(6, \frac{11\pi}{4}\right), \left(6, -\frac{13\pi}{4}\right)$

Exercise 5A (Page 120)

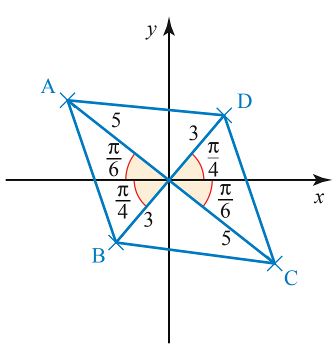
1 (i) $(0, -8)$ (ii) $\left(-4\sqrt{2}, -4\sqrt{2}\right)$

(iii) $\left(4, 4\sqrt{3}\right)$ (iv) $\left(-4\sqrt{3}, 4\right)$

2 (i) $(13, 1.18)$ (ii) $(5, \pi)$

(iii) $\left(2, -\frac{5\pi}{6}\right)$ (iv) $(5, 0.927)$

3

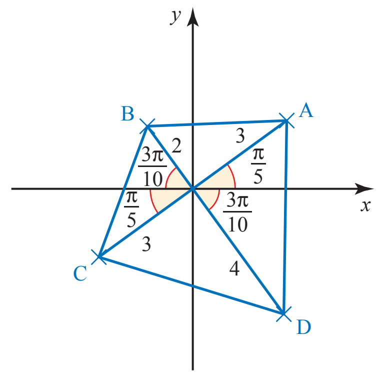


By symmetry the quadrilateral has two pairs of parallel sides.

$\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$ so the diagonals do not meet at right angles and therefore the shape is not a rhombus.

So ABCD is a parallelogram.

4



By symmetry AB and BC are the same length; similarly AD and CD are the same length.

The diagonals of the shape meet at right angles. ABCD is a kite.

5 (i) 4

(ii) $16 < r < 170 \quad \theta = -27$

(iii) (a) $99 < r < 107 \quad 153 < \theta < 171$

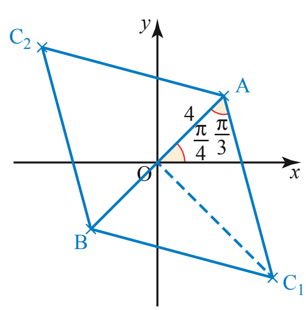
(b) $16 < r < 99 \quad -81 < \theta < -63$ and $107 < r < 162 \quad -81 < \theta < -63$

(c) $162 < r < 170 \quad 45 < \theta < 63$

6 (i) $B\left(4, -\frac{5\pi}{12}\right) \quad C\left(4, \frac{11\pi}{12}\right)$

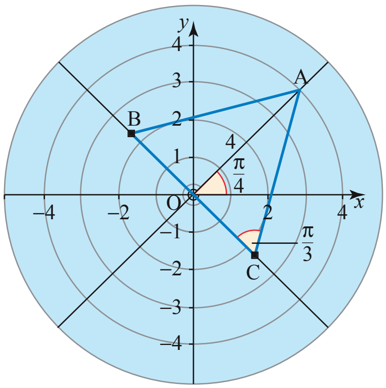
(ii) $B(0, 0)$ and C either $\left(4, -\frac{\pi}{12}\right)$ or $\left(4, \frac{7\pi}{12}\right)$

(iii) The points could be arranged as shown:



If O is the midpoint of AB then B has coordinates $\left(4, -\frac{3\pi}{4}\right)$ and C has coordinates either $\left(4\sqrt{3}, -\frac{\pi}{4}\right)$ or $\left(4\sqrt{3}, \frac{3\pi}{4}\right)$.

Alternatively the points could be arranged as shown below:



If O is the midpoint of BC then B has coordinates $\left(\frac{4\sqrt{3}}{3}, \frac{3\pi}{4}\right)$ and C has coordinates $\left(\frac{4\sqrt{3}}{3}, -\frac{\pi}{4}\right)$.

7 (ii) A(5.39, 0.38) B(8.71, 1.01)
C(8.71, 1.64) D(5.39, 2.27)

(iii) A(5.00, 2.00) B(4.63, 7.38)
C(–0.58, 8.69) D(–3.45, 4.14)

8 (i) $r = 2\sec\theta$

(ii) $\tan\theta = \sqrt{3}$ or $\theta = \pm\frac{\pi}{3}$

(iii) $r = \sqrt{3}$

(iv) $r^2 = 18\operatorname{cosec}2\theta$

(v) $r = \cot\theta\operatorname{cosec}\theta$

(vi) $r^2 = \dfrac{5}{1 - \sin 2\theta}$

9 (i) $x = 5$

(ii) $x^2 + y^2 = 25$

(iii) $y = 4$

(iv) $x^2 + y^2 = 4x$

(v) $(x^2 + y^2)^2 = 2xy$

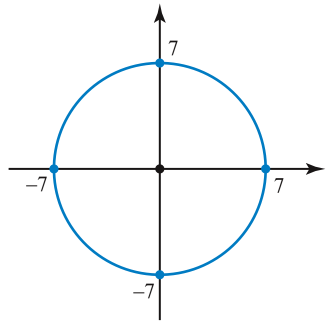
(vi) $x^2 + y^2 = 2\sqrt{(x^2 + y^2)} - 2x$

10 $r = a\sec\theta \Rightarrow r = \dfrac{a}{\cos\theta} \Rightarrow r\cos\theta = a \Rightarrow x = a$

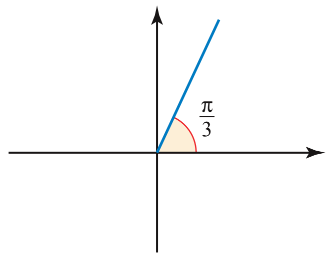
$r = b\operatorname{cosec}\theta \Rightarrow r = \dfrac{b}{\sin\theta} \Rightarrow r\sin\theta = b \Rightarrow y = b$

Activity 5.2 (Page 123)

$r = 7$ is a circle centre (0, 0), radius 7:



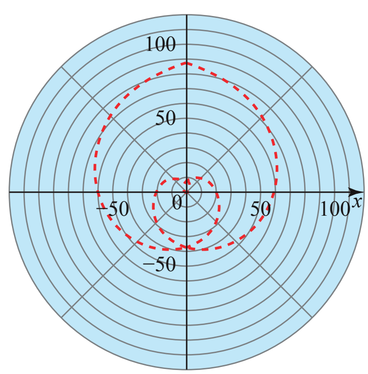
$\theta = \frac{\pi}{3}$ is a half-line starting at the origin making an angle $\frac{\pi}{3}$ with the initial line:



Activity 5.3 (Page 128)

For example

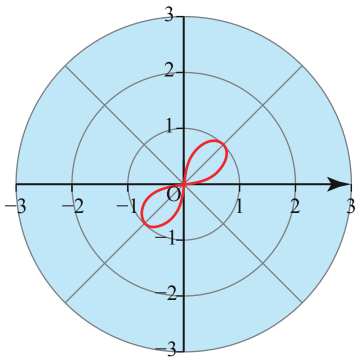
$-\frac{5\pi}{2} \leqslant \theta \leqslant \frac{7\pi}{2}$ using convention 3



Activity 5.4 (Page 129)

$k = 1, \quad n = 2$

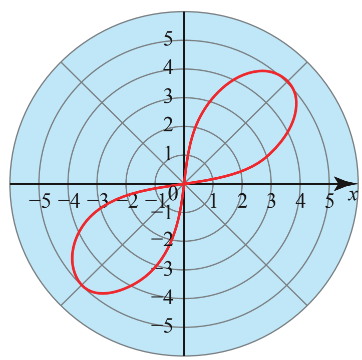
(i)



'Radius' of rhodonea is 1; two petals

$k = 5, \quad n = 2$

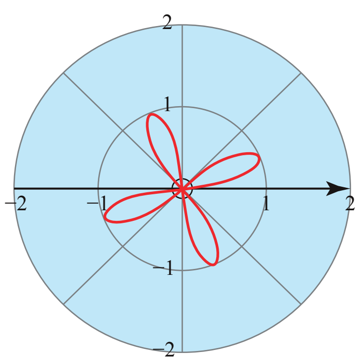
(ii)



'Radius' of rhodonea is 5; two petals

$k = 1, \quad n = 4$

(iii)



'Radius' of rhodonea is 1; four petals

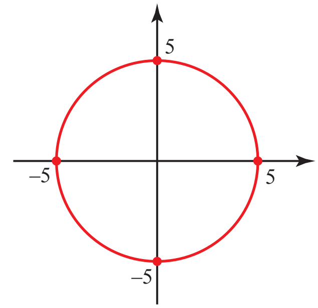
Generally, $r = k\sin(n\theta)$ has 'radius' of k (for $k > 1$).

For $n > 1$ and $r > 0$, the curve has n petals.

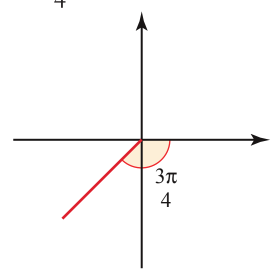
If $r < 0$ included, then the curve has n petals if n is odd and $2n$ petals when n is even. For $k = 1, n = 1$ the curve is a circle of diameter 1 passing through the origin and the point (0, 1), symmetrical about the y-axis.

Exercise 5B (Page 129)

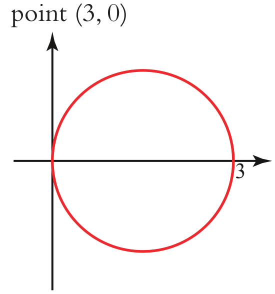
1 (i) Circle centre O, radius 5



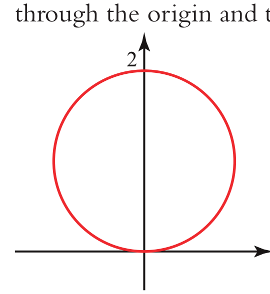
(ii) Half-line from the origin making an angle $-\frac{3\pi}{4}$ with the initial line



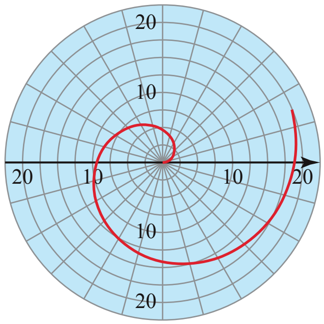
(iii) Circle symmetrical about the x-axis, passing through the origin and the point (3, 0)



(iv) Circle symmetrical about y-axis, passing through the origin and the point (0, 2)



(v) Spiral starting from the origin



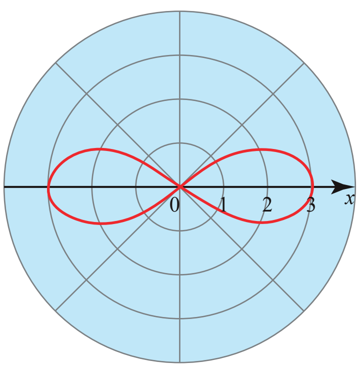
2

$\boldsymbol{\theta}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\boldsymbol{r}$	0	2.07	4	5.66	6.93	7.73	8	7.73	6.93	5.66	4	2.07	0

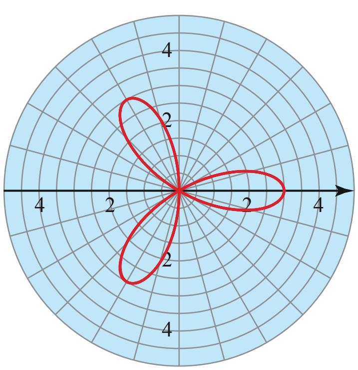
r would be negative

The circle has centre (0, 4), radius 4 so the Cartesian equation is $x^2 + (y - 4)^2 = 16$.

3



$\boldsymbol{r = 3 \cos 2\theta}$



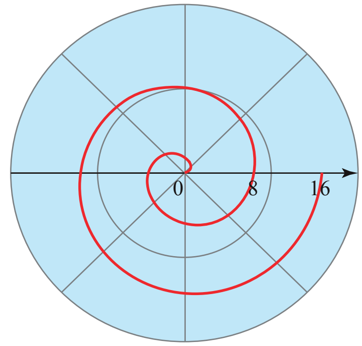
$\boldsymbol{r = 3 \cos 3\theta}$

$r = 3\cos 2\theta$ has two loops and $r = 3\cos 3\theta$ has 3 loops.

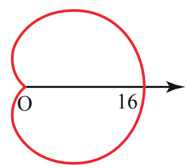
$r = a\cos n\theta$ has n loops

Extension: When negative values of r are included, the loops for $r = a\cos n\theta$ are drawn twice for odd values of n. There are additional loops for $r = a\cos n\theta$ when n is even giving $2n$ loops for $0 \leqslant \theta < 2\pi$.

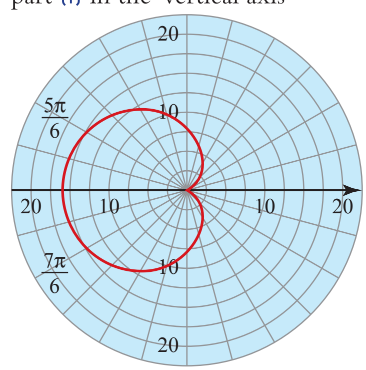
4



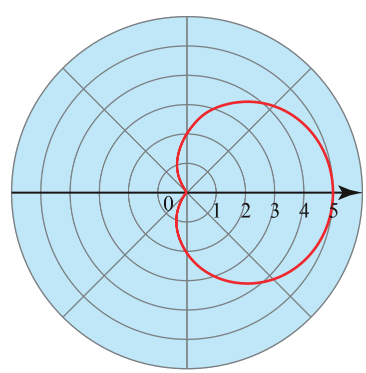
5 (i) The curve has a heart shape, hence the name cardiod.



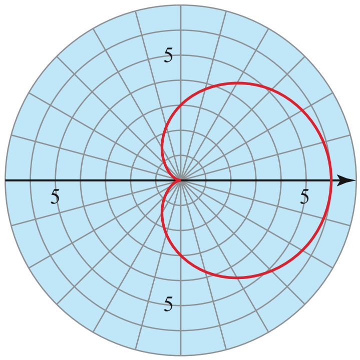
(ii) The curve is a reflection of the curve in part (i) in the 'vertical axis'



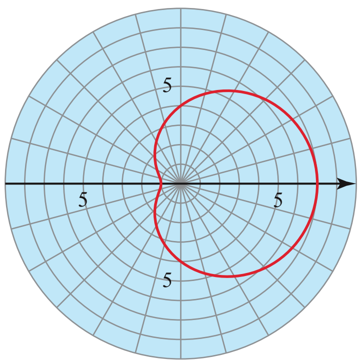
6 (i)



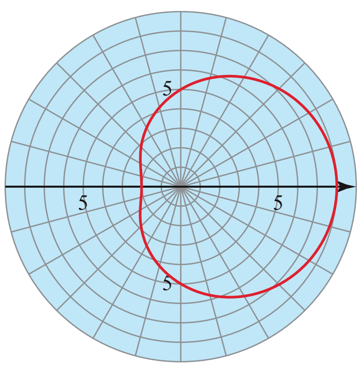
$\boldsymbol{r = 2 + 3 \cos \theta}$



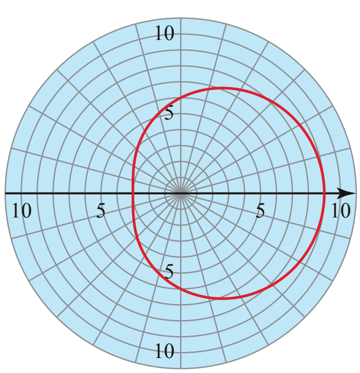
$r = 3 + 3\cos\theta$



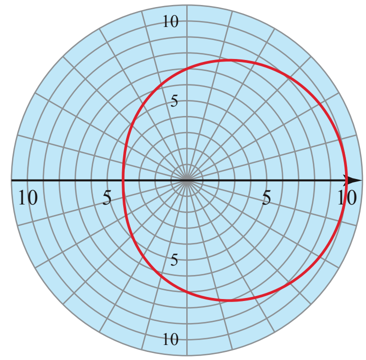
$r = 4 + 3\cos\theta$



$r = 5 + 3\cos\theta$



$r = 6 + 3\cos\theta$



$r = 7 + 3\cos\theta$

(ii) (a) $a \geqslant 2b$ Convex limacon

(b) $2b > a > b$ Limacon has a 'dimple'

(c) $a = b$ Cardioid

(iii) $r \geqslant 0$

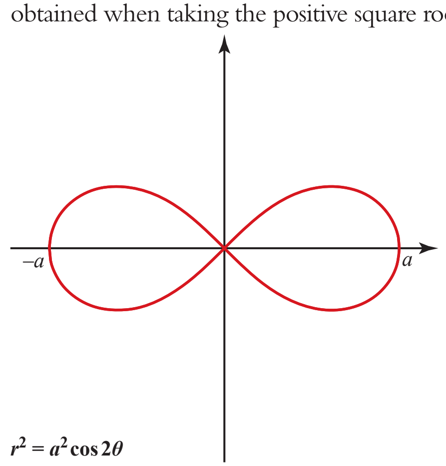
$b\cos\theta \geqslant -a$

$-\arccos\left(-\frac{a}{b}\right) \leqslant \theta \leqslant \arccos\left(-\frac{a}{b}\right)$

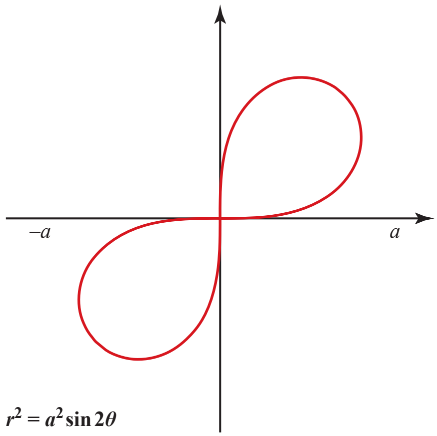
Extension: When $r < 0$ included, limacon has a 'loop' inside for these values of θ.

(iv) The shape is the same but the curves are now symmetrical about the y-axis rather than the x-axis.

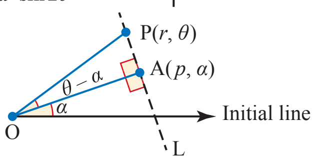
7 Values of r are only defined in the intervals $0 \leqslant \theta \leqslant \frac{\pi}{4}$, $\frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4}$ and $\frac{7\pi}{4} \leqslant \theta \leqslant 2\pi$ (the value of r^2 is negative in the other intervals). Where the curve is defined, one half of each loop is created for each interval of $\frac{\pi}{4}$. Taking negative square roots instead of positive square roots produces the same curve. For example, when $\theta = 0$, $r = -a$ when taking the negative square root; this is equivalent to the point (π, a) obtained when taking the positive square root.



$r^2 = a^2\cos 2\theta$



8 (i)



Let (r, θ) be a point on the line L that is perpendicualr to OA. In triangle OPA,

$\cos(\theta - \alpha) = \frac{p}{r} \Rightarrow r\cos(\theta - \alpha) = p$

(ii) $r\cos(\theta - \alpha) = p$

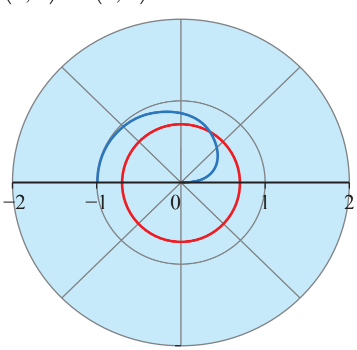
$\Rightarrow r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$

9 Intersection at $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$;

C_1: Circle centre at the pole and radius $\frac{1}{\sqrt{2}}$

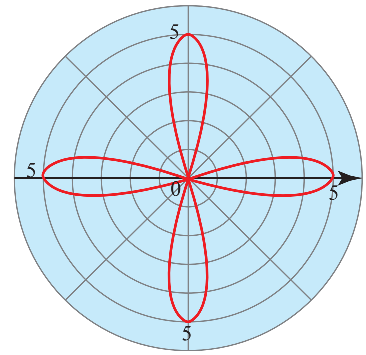
C_2: Curve approx. correct orientation from $(0, 0)$ to $(1, \pi)$



Exercise 5C (Page 132)

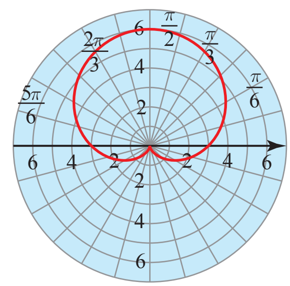
1 Area is 25π

2 (i)



(ii) $\frac{25\pi}{16}$

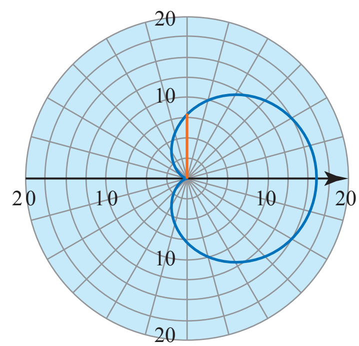
3 (i)



(ii) $\frac{27\pi}{2}$

4 $\frac{64\pi}{3}$

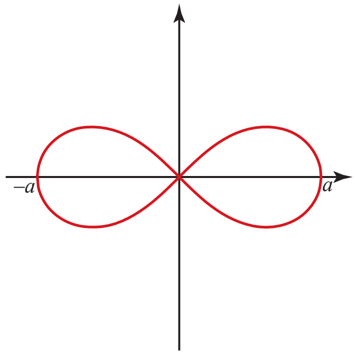
5



Larger part $= 24\pi + 64$

Smaller part $= 24\pi - 64$

6



One loop has area $\frac{1}{2}a^2$.

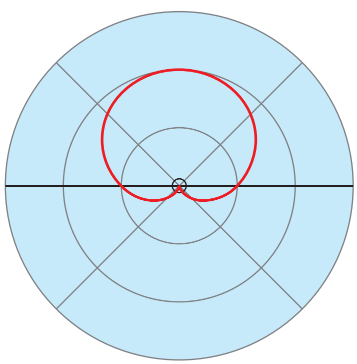
7 $A = \frac{a^2}{4k}\left(e^{\frac{k\pi}{2}} - 1\right)$, $B = \frac{a^2}{4k}e^{4k\pi}\left(e^{\frac{k\pi}{2}} - 1\right)$,

$C = \frac{a^2}{4k}e^{8k\pi}\left(e^{\frac{k\pi}{2}} - 1\right)$

Common ratio $e^{4k\pi}$

8 1.15

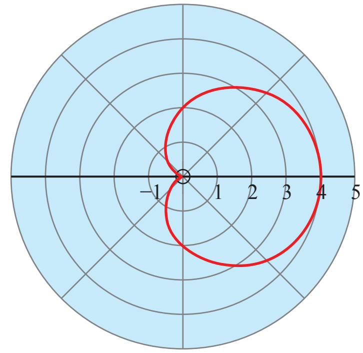
9



Sketch should show a cardioid through $(a, 0)$, $\left(2a, \frac{\pi}{2}\right)$, (a, π) and $\left(0, \frac{3\pi}{2}\right)$;

$a^2\left(\frac{\pi}{4} + 1 + \frac{1}{8}\sqrt{3}\right)$

10



Sketch should show upper half of cardioid through $(4, 0)$ and $(0, \pi)$;

3π;

4.712, 4.713

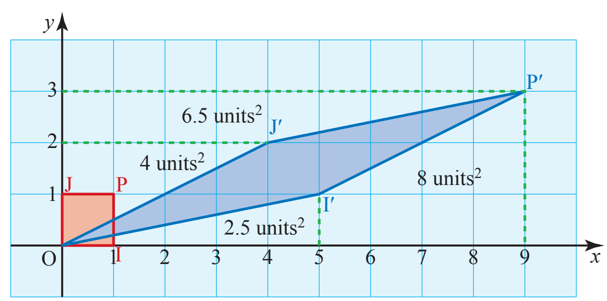
Chapter 6

? (Page 134)

The triangles are all congruent to each other.
256 yellow triangles make up the purple triangle.

Activity 6.1 (Page 135)

The diagram shows the image of the unit square OIPJ under the transformation with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.



The point I(1, 0) is transformed to the point I' (a, c); the point J(0, 1) is transformed to the point J' (b, d).
P' has coordinates $(a + b, c + d)$.

The area of the parallelogram is given by the area of the whole rectangle minus the area of the rectangles and triangles.

Area of first rectangle $= b \times c$

Area of first triangle $= \frac{1}{2} \times b \times d$

Area of second triangle $= \frac{1}{2} \times a \times c$

Area of whole rectangle $= (a + b) \times (c + d)$

Therefore the area of the parallelogram is

$$(a + b) \times (c + d) - 2\left(bc + \frac{1}{2}bd + \frac{1}{2}ac\right) = ad - bc.$$

? (Page 137)

(i) A rotation does not reverse the order of the vertices, e.g. for $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

det $\mathbf{A} = 1$, which is positive.

(ii) A reflection reverses the order of the vertices, e.g. for $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

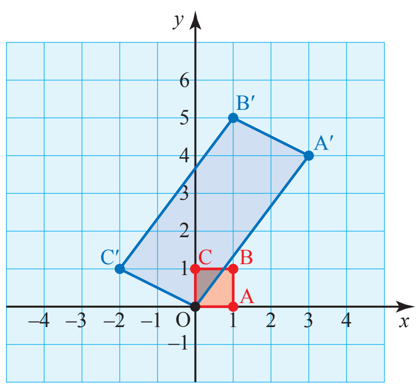
det $\mathbf{B} = -1$, which is negative.

(iii) An enlargement does not reverse the order of the vertices, e.g. for $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,

det $\mathbf{C} = 4$, which is positive

Exercise 6A (Page 139)

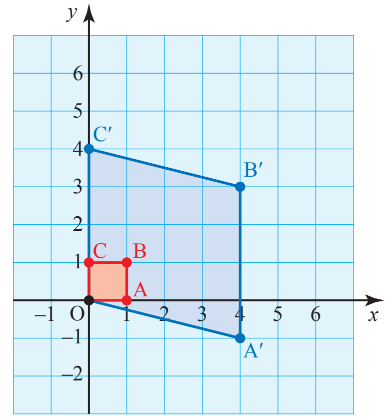
1 (i) (a)



(b) Area of parallelogram = 11

(c) 11

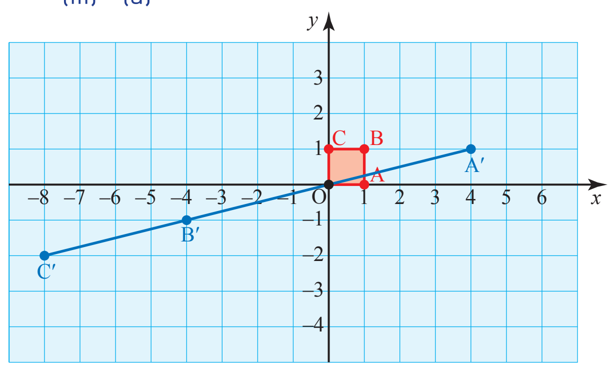
(ii) (a)



(b) Area of parallelogram = 16

(c) 16

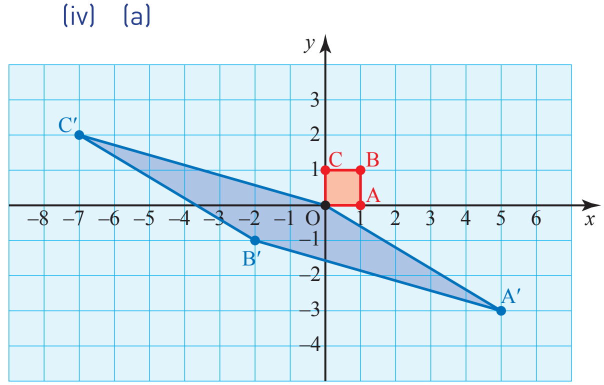
(iii) (a)



(b) area of parallelogram = 0

(c) 0

(iv) (a)



(b) area of parallelogram = 11

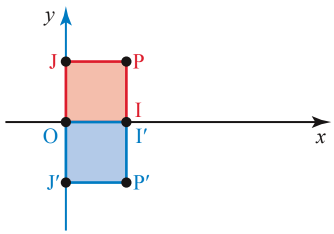
(c) –11

2 $x = 2, x = 6$

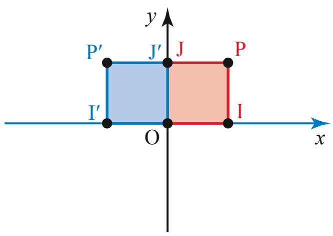
3 (i) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \mathbf{D} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

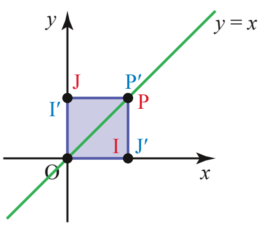
(iii) **A**



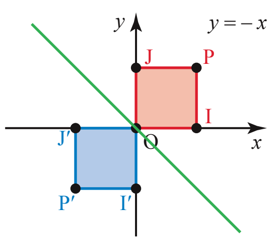
B



C



D



4 66 cm²

5 $ad = 1$

6 (i) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

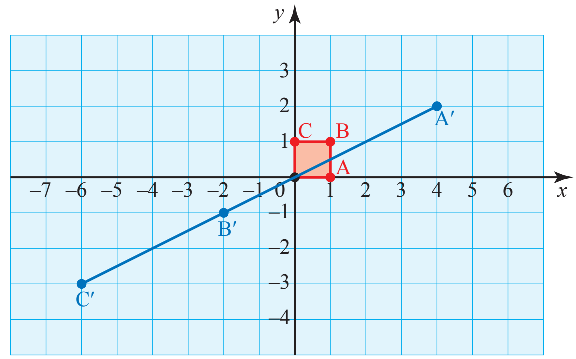
(ii) determinant = 1 so area is preserved

7 (i) det $\mathbf{M} = -2$, $\quad$ det $\mathbf{N} = 7$

(ii) $\mathbf{MN} = \begin{pmatrix} 9 & 13 \\ 8 & 10 \end{pmatrix}$, det $\mathbf{MN} = -14$

and $-14 = -2 \times 7$

8 (i)



(ii) The image of all points lie on the line $y = \frac{1}{2}x$. The determinant of the matrix is zero, which shows that the image will have zero area.

9 (i) $\begin{pmatrix} 5p - 10q \\ -p + 2q \end{pmatrix}$

(ii) $y = -\frac{1}{5}x$

(iii) det $\mathbf{N} = 0$ and so the image has zero area

10 (i) $\mathbf{T} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$,

det $\mathbf{T} = (1 \times 6) - (3 \times 2) = 0$

(iii) (3, 9)

12 (i) $y = 3x - 3s + t$

(ii) $\mathrm{P'}\left(\frac{9}{8}s - \frac{3}{8}t,\ \frac{3}{8}s - \frac{1}{8}t\right)$

(iii) $\begin{pmatrix} \frac{9}{8} & -\frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{pmatrix}$, which has determinant

$\left(\frac{9}{8} \times -\frac{1}{8}\right) - \left(\frac{3}{8} \times -\frac{3}{8}\right) = 0$

Activity 6.2 (Page 141)

(i) $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(ii) $\mathbf{P}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iii) Reflecting an object in the x-axis twice takes it back to the starting position and so the final image is the same as the original object. Hence the matrix for the combined transformation is **I**.

Activity 6.3 (Page 142)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

To turn this into the identity matrix it would need to be divided by $ad - bc$, which is the value $|\mathbf{M}|$.

Therefore $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Activity 6.4 (Page 143)

(i) $\mathbf{AA}^{-1} = \frac{1}{4}\begin{pmatrix} 11 & 3 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \mathbf{I}$

$\mathbf{A}^{-1}\mathbf{A} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}\begin{pmatrix} 11 & 3 \\ 6 & 2 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \mathbf{I}$

(ii) $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\mathbf{MM}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \dfrac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ab \\ cd-dc & -cb+ad \end{pmatrix}$

$= \dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

$\mathbf{M}^{-1}\mathbf{M} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \dfrac{1}{ad-bc}\begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix}$

$= \dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

? (Page 143)

First reverse the reflection by using the transformation with the inverse matrix of the reflection. Secondly, reverse the rotation by using the transformation with the inverse matrix of the rotation.

$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$

Exercise 6B (Page 144)

1 (i) $(10, -6)$

(ii) $-\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix}$

(iii) $(1, 2)$

2 (i) non-singular, $\frac{1}{24}\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$

(ii) singular

(iii) non-singular, $\frac{1}{112}\begin{pmatrix} 11 & -3 \\ -3 & 11 \end{pmatrix}$

(iv) singular

(v) singular

(vi) singular

(vii) non-singular, $\dfrac{1}{16(1-ab)}\begin{pmatrix} -8 & -4a \\ -4b & -2 \end{pmatrix}$

provided $ab \neq 1$

3 (i) $\frac{1}{3}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$ (ii) $\frac{1}{2}\begin{pmatrix} -1 & -5 \\ 2 & 8 \end{pmatrix}$

(iii) $\begin{pmatrix} 28 & 19 \\ 10 & 7 \end{pmatrix}$ (iv) $\begin{pmatrix} 50 & 63 \\ -12 & -15 \end{pmatrix}$

(v) $\frac{1}{6}\begin{pmatrix} 7 & -19 \\ -10 & 28 \end{pmatrix}$ (vi) $\frac{1}{6}\begin{pmatrix} -15 & -63 \\ 12 & 50 \end{pmatrix}$

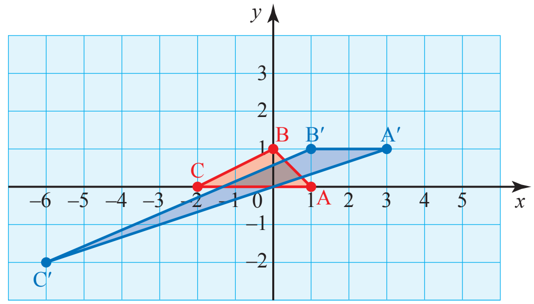
(vii) $\frac{1}{6}\begin{pmatrix} -15 & -63 \\ 12 & 50 \end{pmatrix}$ (viii) $\frac{1}{6}\begin{pmatrix} 7 & -19 \\ -10 & 28 \end{pmatrix}$

4 (i) $\frac{1}{8}\begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$

5 $k = 2$ or $k = 3$

6 $\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -3 & 4 \end{pmatrix}$

7 (i) $(3, 1)$, $(1, 1)$ and $(-6, -2)$



(ii) ratio of area T' to T is 3 : 1.5 or 2 : 1

The determinant of the matrix **M** = 2 so the area is doubled.

(iii) $\mathbf{M}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$

8 (ii) $\mathbf{M}^n = (a+d)^{n-1}\mathbf{M}$

9 (iii) $\begin{pmatrix} 0 & \frac{1}{5} \\ \frac{1}{6} & -\frac{11}{10} \end{pmatrix}$ (iv) $\begin{pmatrix} 33 & 6 \\ 5 & 0 \end{pmatrix}$

Activity 6.5 (Page 146)

$\det \mathbf{A} = -17 \quad \mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{17} & -\frac{2}{17} & \frac{5}{17} \\ -\frac{8}{17} & \frac{1}{17} & \frac{6}{17} \\ \frac{4}{17} & \frac{8}{17} & -\frac{3}{17} \end{pmatrix}$

B is singular.

$\mathbf{C}^{-1} = \begin{pmatrix} -\frac{1}{15} & -\frac{2}{3} & \frac{1}{5} \\ \frac{2}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{15} & \frac{1}{3} & \frac{1}{5} \end{pmatrix}$

$\mathbf{D}^{-1} = \begin{pmatrix} -1 & -3 & \frac{4}{3} \\ 1 & 2 & -\frac{2}{3} \\ 1 & 3 & -1 \end{pmatrix}$

Exercise 6C (Page 151)

1 (i) (a) 5 (b) 5

(ii) (a) –5 (b) –5

Interchanging the rows and columns has not changed the determinant.

(iii) (a) 0 (b) 0

If a matrix has a repeated row or column the determinant will be zero.

2 (i) $\frac{1}{3}\begin{pmatrix} 3 & 0 & 6 \\ -4 & 2 & 3 \\ 2 & -1 & 0 \end{pmatrix}$

(ii) Matrix is singular

(iii) $\begin{pmatrix} -0.06 & -0.1 & -0.1 \\ 0.92 & 0.2 & 0.7 \\ 0.66 & 0.1 & 0.6 \end{pmatrix}$

(iv) $\frac{1}{21}\begin{pmatrix} 34 & 11 & 32 \\ 9 & 6 & 6 \\ -38 & -16 & -37 \end{pmatrix}$

3 $\mathbf{M}^{-1} = \frac{1}{28-10k}\begin{pmatrix} 4 & -10 & 12 \\ -(4k-8) & 8 & -(4+2k) \\ -k & 7 & -3k \end{pmatrix}$

$k = 2.8$

4 (i) The columns of the matrix have been moved one place to the right, with the final column moving to replace the first. This is called **cyclical interchange** of the columns.

(ii) det **A** = det **B** = det **C** = –26

Cyclical interchange of the columns leaves the determinant unchanged.

5 $x = \frac{-1 \pm \sqrt{41}}{2}$

6 $x = 1, x = 4$

7 $1 < k < 5$

8 (ii) $\mathbf{P}^{-1} = \begin{pmatrix} -\frac{1}{9} & \frac{1}{6} & -\frac{4}{9} \\ \frac{2}{9} & \frac{1}{6} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$

$\mathbf{Q}^{-1} = \begin{pmatrix} \frac{3}{2} & -4 & \frac{1}{2} \\ 1 & -2 & 0 \\ -\frac{3}{2} & 5 & -\frac{1}{2} \end{pmatrix}$

$$(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} = \begin{pmatrix} -\frac{11}{9} & -\frac{1}{6} & -\frac{7}{18} \\ -\frac{5}{9} & -\frac{1}{6} & -\frac{2}{9} \\ \frac{13}{9} & \frac{1}{3} & \frac{5}{18} \end{pmatrix}$$

9 (ii) Multiplying only the first column by k equates to a stretch of scale factor k in one direction, so only multiplies the volume by k.

(iii) Multiplying any column by k multplies the determinant by k.

10 (i) $10 \times 43 = 430$

(ii) $4 \times 5 \times -7 \times 43 = -6020$

(iii) $x \times 2 \times y \times 43 = 86xy$

(iv) $x^4 \times \frac{1}{2x} \times 4y \times 43 = 86x^3y$

Chapter 7

? (Page 155)

Grimshaw architects designed the Eden project using as their starting point the geodesic design of Buckminster Fuller. Fuller used spherical geometry to enclose the most space with the least material and believed that this would put an end to the dominance of the right angle in architecture.

? (Page 156)

A three-legged stool is the more stable. Three points, such as the ends of the legs, define a plane but a fourth will not, in general, be in the same plane. So the ends of the legs of a three-legged stool lie in a plane but those of a four-legged stool need not. The four-legged stool will rest on three legs but could rock on to a different three.

Three points on the plane are sufficient to specify a plane. So too would two lines in the plane, or one point on the plane and the direction of the normal vector. These ideas are explored in this chapter.

? (Page 158)

(i) 90° with all lines.

(ii) No, so long as the pencil remains perpendicular to the table.

Activity 7.1 (Page 164)

Repeat the work in Example 7.6 replacing (7, 5, 3) by (α, β, γ), so 7 by α, 5 by β and 3 by γ, and (3, 2, 1) by (n_1, n_2, n_3) and 6 by d.

Exercise 7A (Page 165)

1 (i) Parallel, line in plane

(ii) Parallel, line not in plane

(iii) Not parallel

(iv) Parallel, line in plane

(v) Not parallel

(vi) Parallel, line not in plane

2 (i) $\overrightarrow{LM} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}; \overrightarrow{LN} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

(iii) $x - 4y - 3z = -2$

3 (iii) B

4 (i) Accept any points that satisfy the equations.

(iii) Three points define a plane.

(iv) (1, 0, −1)

5 (i) (0, 1, 3)

(ii) (1, 1, 1)

(iii) (8, 4, 2)

(iv) (0, 0, 0)

(v) (11, 19, −10)

6 (i) (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) (1, 3, 1)

(c) $\sqrt{6}$

(ii) (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$

(b) (1, 0.5, −1.5)

(c) 3.08

(iii) (a) $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b) (0, 1, 3)

(c) 3

(iv) (a) $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$

(b) (2, 1, 0): A is in the plane

(c) 0

(v) (a) $\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) (2, 2, 2)

(c) $\sqrt{12}$

7 (i) $x + 2y + 3z = 25$

(ii) $206 = 150 + 56$

(iii) W is in the plane; $\overrightarrow{UW} \cdot \overrightarrow{UV} = 0$

8 (i) $\mathbf{r} = \begin{pmatrix} 13 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

(ii) (4, 2, 6)

(iii) 11.2

9 (i) 4.1°

(ii) 32.3°

(iii) 35.6°

10 (ii) $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \overrightarrow{AC} = \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}$; in both cases the scalar product = 0

(iii) 132.9°

(iv) 8.08

11 (i) 5, $\sqrt{89}$

(ii) 62.2°

(iii) 20.9

(iv) (4, 6, −3)

12 (i) PQ: $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$;

XY: $\mathbf{r} = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(iii) Yes

(iv) Yes, (1, 4, 6)

13 (i) $b = -2, c = 3$

14 (ii) $6x + y - 8z = 6$

15 (i) $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$

(iii) $7x - 11y + 8z = 0$

16 (i) $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(ii) 72.2°

(iii) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(6\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

? (Page 172)

π_3 is parallel to π_1 and π_2 (the common line is at infinity).

Exercise 7B (Page 173)

1 (i) $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 15 \\ 27 \\ 7 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$

(iii) $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ 15 \\ 13 \end{pmatrix}$

(iv) $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 4 \\ 21 \end{pmatrix}$

2 (i) 56.5°

(ii) 80.0°

(iii) 24.9°

(iv) 63.5°

3 (i) $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$

4 $41x - 19y + 26z = 33$

5 $x + 3y - z = -8$

6 $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 21 \\ 4 \\ 11 \end{pmatrix}$

7 $60x + 11y + 100z = 900$;
$60x - 11y - 100z = -300$;
$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} + t \begin{pmatrix} 0 \\ 100 \\ -11 \end{pmatrix}$; 6.3°

8 (i) $x + 3z = -800$
(ii) Normal is approx. 18.4° to the horizontal
(iii) $14x - 15y + 3450z = 15\,950$
(iv) $x = 15\lambda$,
$y = -1136\lambda - 62\,396.7$,
$z = -5\lambda - 266.7$
(v) 62 km (assuming seam is sufficiently extensive)

9 (i) $a = -2$
(ii) 3

10 (i) $4x + 2y + z = 8$
(ii) 77.4°

11 (i) 57.7°
(ii) $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$

12 (i) $2x - 3y + 6z = 2$
(ii) 2
(iii) $\mathbf{r} = \lambda(6\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

13 $\overrightarrow{OP} = \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$

? (Page 178)

Use the scalar product to check that the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

$$(\mathbf{a} \times \mathbf{b}).\mathbf{a} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$= (24 \times 3) + (-1 \times 2) + (-14 \times 5) = 0$$

$$(\mathbf{a} \times \mathbf{b}).\mathbf{b} = \begin{pmatrix} 24 \\ -1 \\ -14 \end{pmatrix} . \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$= (24 \times 1) + (-1 \times -4) + (-14 \times 2) = 0$$

Activity 7.2 (Page 179)

$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ $\quad \mathbf{j} \times \mathbf{j} = 0$ $\quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$

$\mathbf{k} \times \mathbf{i} = \mathbf{j}$ $\quad \mathbf{k} \times \mathbf{k} = 0$ $\quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$

Exercise 7C (Page 181)

1 (i) $\begin{pmatrix} -23 \\ 13 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 37 \\ 41 \\ 19 \end{pmatrix}$

(iii) $\begin{pmatrix} -8 \\ 34 \\ 27 \end{pmatrix}$ (iv) $\begin{pmatrix} 21 \\ -29 \\ 9 \end{pmatrix}$

2 (i) $\begin{pmatrix} 5 \\ 19 \\ -2 \end{pmatrix}$ (ii) $\begin{pmatrix} 14 \\ -62 \\ -9 \end{pmatrix}$

(iii) $\begin{pmatrix} -3 \\ -2 \\ 3 \end{pmatrix}$ (iv) $\begin{pmatrix} -18 \\ 57 \\ 47 \end{pmatrix}$

3 (i) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 \\ 12 \\ 15 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

(iii) $x + 4y + 5z = 7$

4 $\frac{1}{\sqrt{635}} \begin{pmatrix} 19 \\ 15 \\ -7 \end{pmatrix}$

5 $\sqrt{74}$

6 (i) $5x + 4y - 8z = 5$
(ii) $24x + y - 29z = 1$
(iii) $19x + 40y + 3z = 188$
(iv) $30x - 29y - 24z = 86$

7 (i) $-8\mathbf{j}$ (ii) $6\mathbf{j}-4\mathbf{k}$

(iii) $2\mathbf{i}-12\mathbf{j}$ (iv) $6\mathbf{i}+14\mathbf{j}-2\mathbf{k}$

9 (i) $\frac{1}{2}\sqrt{717}$ (ii) $\sqrt{717}$

10 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \mathbf{i}+3\mathbf{j}-\mathbf{k}$

Equation of Π_1: $x+3y-z=12$

$$\cos\theta = \left|\frac{2-3-1}{\sqrt{11}\sqrt{6}}\right|$$

$$= \frac{2}{\sqrt{66}} \Rightarrow \theta = 75.7° \text{ or } 1.32 \text{ rad}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2\mathbf{i}+3\mathbf{j}-7\mathbf{k}$$

Points on both planes is, e. g. (6, 2, 0)

$\mathbf{r} = 6\mathbf{i}+2\mathbf{j}+t(2\mathbf{i}-3\mathbf{j}-7\mathbf{k})$ or equivalent

Activity 7.3 (Page 185)

(i) $A = \left(0, -\frac{c}{b}\right)$, $A' = \left(0, -\frac{c}{b}, 0\right)$

(ii) $y = -\frac{a}{b}x - \frac{c}{b}$, $d=b$, $e=-a$, $f=0$

Activity 7.4 (Page 189)

Because it is perpendicular to both l_1 and l_2.

Exercise 7D (Page 190)

1 (i) $\sqrt{29}$ (ii) 7 (iii) $2\sqrt{26}$

2 (i) 13 (ii) $3\sqrt{5}$ (iii) 1

3 (i) 5 (ii) 5 (iii) $5\sqrt{2}$

4 (i) $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ (ii) $\sqrt{3}$

5 (ii) $\frac{1}{2}\sqrt{38}$

6 (i) 4, the lines are skew

(ii) 0.4, the lines are skew

(iii) 0, the lines intersect

(iv) $\frac{\sqrt{77}}{\sqrt{6}}$, the lines are parallel

7 (i) $2\sqrt{69}$ (ii) M(−3, 2, 6)

8 (i) $\frac{2\sqrt{5}}{15}$

(ii) $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$

(iii) $M\left(\frac{82}{45}, \frac{10}{45}, -\frac{229}{45}\right)$

9 (i) AB: $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and

CD: $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix}$ or

equivalent; 6

(ii) 3.08 m – No, the cable is not long enough

10 (ii) $\frac{1}{15}(10k+105)$

(iii) $k = \frac{-21}{2}$

(iv) $\mathbf{r} = \begin{pmatrix} 4 \\ 12 \\ 5 \end{pmatrix} + \alpha\begin{pmatrix} 2 \\ 10 \\ 11 \end{pmatrix}$ or equivalent

11 $\frac{16}{3}$, 36.7°

12 $m=2$ or $\frac{2}{19}$, but m is an integer so $m=2$

shortest distance $\sqrt{\frac{18}{17}}$